Tokové polynómy k-pólov

Diplomová práca

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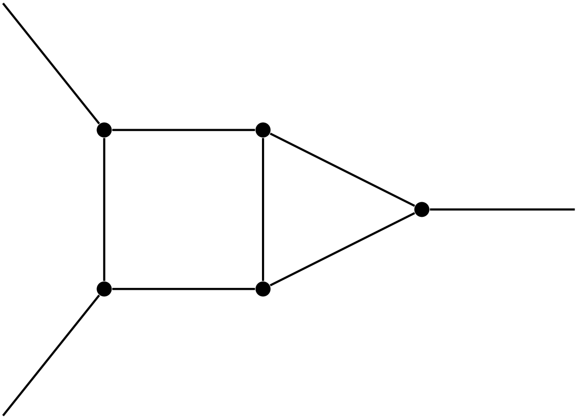
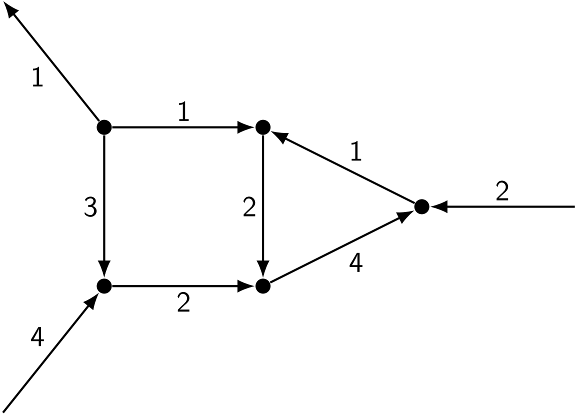
k-pól



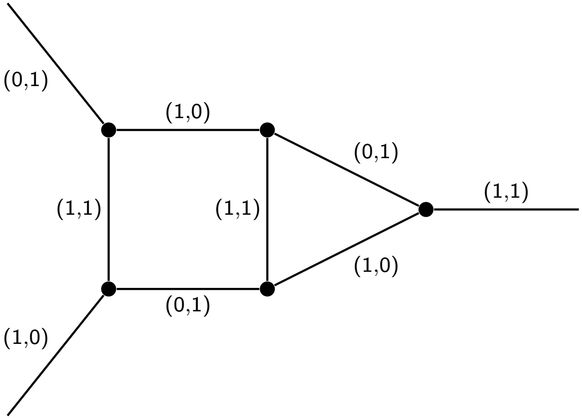
Obr.: Príklad kubického 3-pólu.

Nikde-nulový tok nad grupou



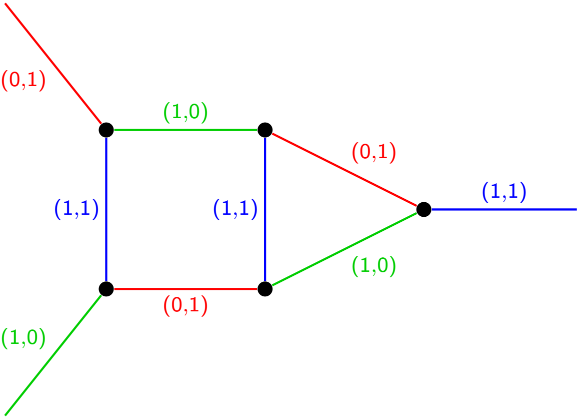
Obr.: Príklad nikde-nulového $\mathbb{Z}_5$-toku v 3-póle.

$(\mathbb{Z}_2 \times \mathbb{Z}_2)$-tok



Obr.: Príklad nikde-nulového $(\mathbb{Z}_2 \times \mathbb{Z}_2)$-toku v 3-póle. Na orientácii hrán nezáleží – v $(\mathbb{Z}_2 \times \mathbb{Z}_2, +)$ je každý prvok involúciou.

$(\mathbb{Z}_2 \times \mathbb{Z}_2)$-tok



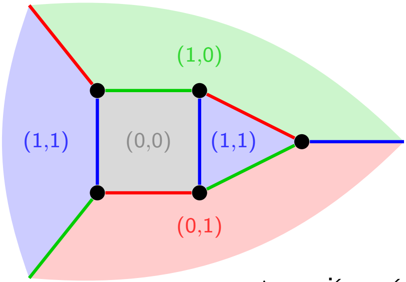
Obr.: Nikde-nulový $(\mathbb{Z}_2 \times \mathbb{Z}_2)$-tok v kubickom k-póle tvorí hranové 3-farbenie.

Tokovo-farbiaca dualita

Planárny k-pól G:

m-farbenie oblastí $\Leftrightarrow$ nikde-nulový m-tok

veta o 4 farbách $\Leftrightarrow$ nikde-nulový $(\mathbb{Z}_2 \times \mathbb{Z}_2)$-tok

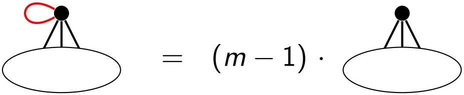


+ zaujímavé sú kubické grafy

Počet tokov $f(G)$

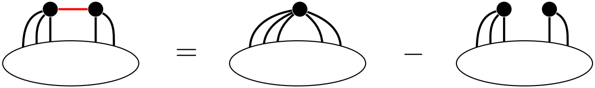
- ak e je slučka v G:

$$\boldsymbol{f(G) = (m-1) \cdot f(G-e)}$$

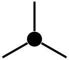


- inak:

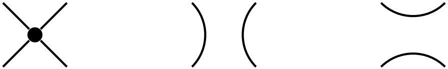
$$\boldsymbol{f(G) = f(G\,/\,e) - f(G-e)}$$



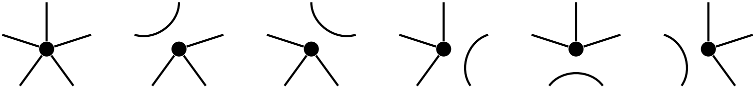
Základné k-póly = bez vnútorných hrán



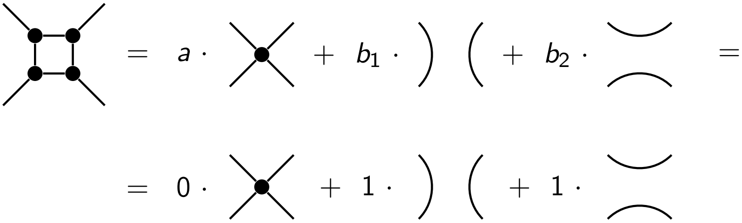
Obr.: Planárny 3-pól bez vnútorných hrán.



Obr.: Planárne 4-póly bez vnútorných hrán.

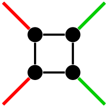
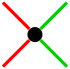
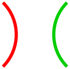
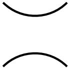


Obr.: Planárne 5-póly bez vnútorných hrán.



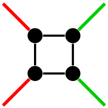
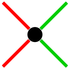
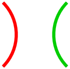
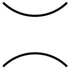
= a · + b1 · + b2 · =
= 0 · + 1 · + 1 ·

$f(G)$ ako kombinácia základných k-pólov

 $= a \cdot$  $+ \; b_1 \cdot$  $+ \; b_2 \cdot$  $=$

$$= 0 \cdot 1 + 1 \cdot 1 + 1 \cdot 0$$

$f(G)$ ako kombinácia základných k-pólov

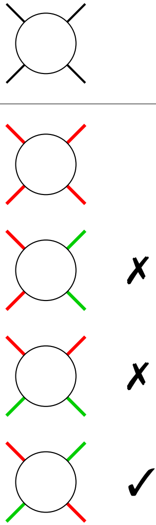
 $=\ a\cdot$  $+\ b_1\cdot$  $+\ b_2\cdot$  $=$

$$= 0\cdot 1 + 1\cdot 1 + 1\cdot 0$$

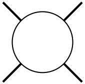
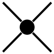
Na $\mathbb{Z}_5$:

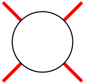
Kochol (2004). Reduction of the 5-Flow Conjecture to cyclically 6-edge-connected snarks.
Kochol (2005). Decomposition formulas for the flow polynomial.
Kochol (2006). Restrictions On Smallest Counterexamples To The 5-Flow Conjecture.
Kochol (2010). Smallest counterexample to the 5-flow conjecture has girth at least eleven.

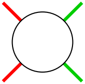
Motivácia

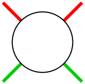


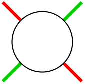
Motivácia

 $= \quad a \cdot$  $+ \quad b_1 \cdot \big) \; \big(+ \quad b_2 \cdot$ 

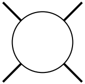
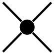
 $a + b_1 + b_2 \geq 0$

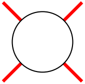
 ✗ $a + b_1 = 0$

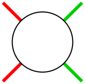
 ✗ $a + b_2 = 0$

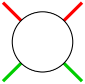
 ✓ $a > 0$

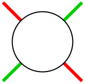
Motivácia

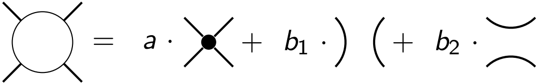
 $= a \cdot$  $+ \ b_1 \cdot) \ (+ \ b_2 \cdot$ 

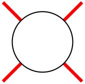
 $a + b_1 + b_2 \geq 0$

 ✗ $a + b_1 = 0$ $\Rightarrow b_1 = -a$

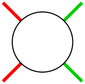
 ✗ $a + b_2 = 0$ $\Rightarrow b_2 = -a$

 ✓ $a > 0$ $\Rightarrow a + b_1 + b_2 < 0$ **SPOR**

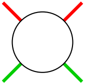




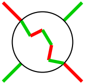
$a + b_1 + b_2 \geq 0$



✗ $a + b_1 = 0 \quad \Rightarrow b_1 = -a$



✗ $a + b_2 = 0 \quad \Rightarrow b_2 = -a$



✓ $a > 0 \quad \Rightarrow a + b_1 + b_2 < 0$ **SPOR**

Naša práca

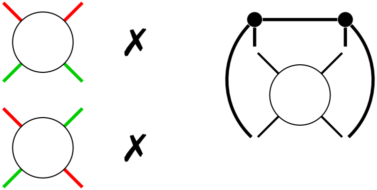
Koeficienty a farbenia

k	koeficienty	farbenia
2	1	1
3	1	1
4	3	4
5	6	10
6	15	31
7	36	91
8	91	274
$\dots$	$\Theta(3^k/k^{3/2})$	$\Theta(3^k)$

dôkaz ostrej nerovnosti pre $k \geq 4$

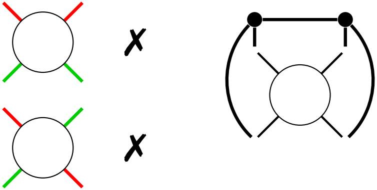
$\rightarrow$ pre planárne grafy sú koeficienty efektívnym nástrojom na štúdium farbení (pre neplanárne však nie)

Indikátor (p, r)-súvislosti



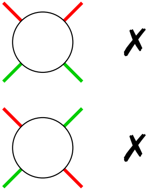
$$2a + b_2 = 0$$

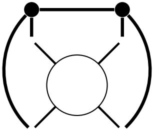
Indikátor (p, r)-súvislosti



$$2a + b_2 = 0$$
$$\Updownarrow$$
G je $(2, 2)$-nesúvislý

Indikátor (p, r)-súvislosti



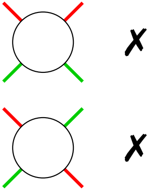


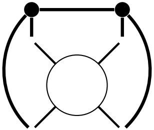
$$2a + b_2 = 0$$
$$\Updownarrow$$
$$G \text{ je } (2,2)\text{-nesúvislý}$$

Veta

G je (p, r)-súvislý práve vtedy, keď súčet počtov (p, r)-nevyvážených farbení je nenulový.

Indikátor (p, r)-súvislosti





$$2a + b_2 = 0$$
$$\Updownarrow$$
G je $(2, 2)$-nesúvislý

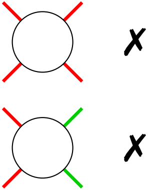
Veta

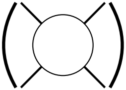
G je (p, r)-súvislý práve vtedy, keď súčet počtov (p, r)-nevyvážených farbení je nenulový.

Veta

G je (p, r)-súvislý práve vtedy, keď aspoň jeden jeho (p, r)-súvislý koeficient je nenulový.

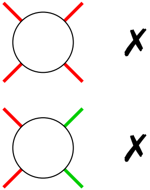
Indikátor (p, r)-2-súvislosti

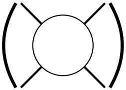




$$2a + 2b_1 + b_2 = 0$$

Indikátor (p, r)-2-súvislosti





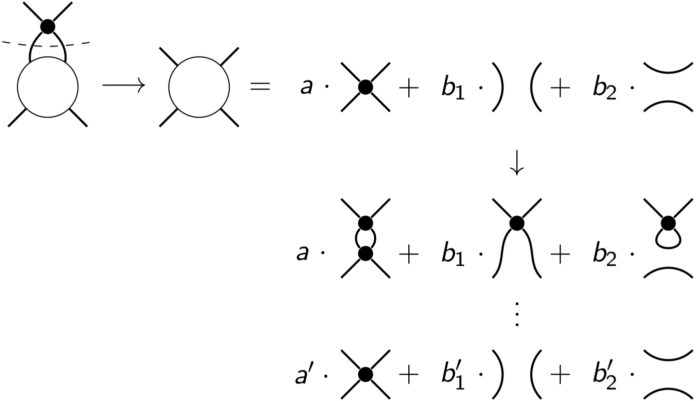
$$2a + 2b_1 + b_2 = 0$$
$$\Updownarrow$$
$$G \text{ má } (2,2)\text{-most}$$

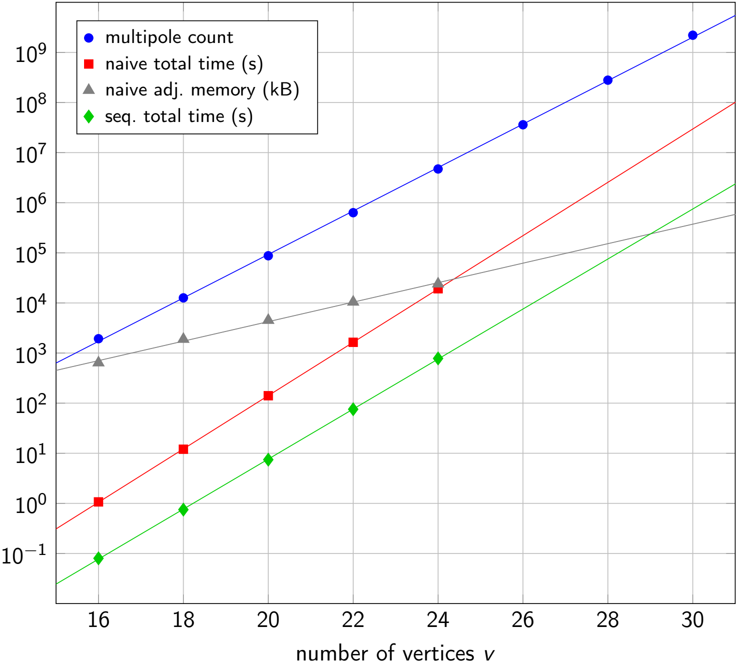
Veta

G je (p, r)-2-súvislý práve vtedy, keď súčet počtov (p, r)-vyvážených farbení je nenulový.

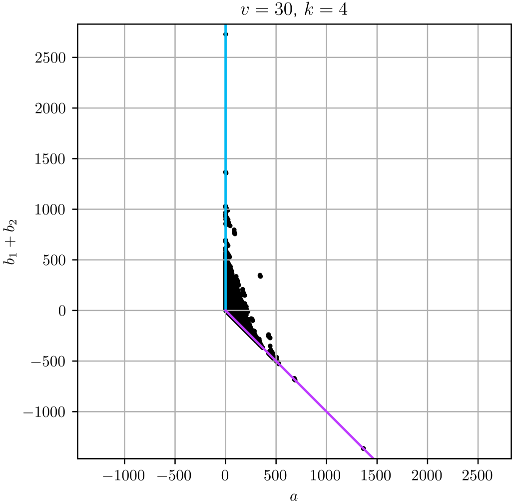
Algoritmus výpočtu koeficientov

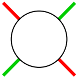
- NAIVE: rekurzia, memoizácia, orezávanie
- SEQ:



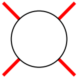

multipole count
naive total time (s)
naive adj. memory (kB)
seq. total time (s)
10^9
10^8
10^7
10^6
10^5
10^4
10^3
10^2
10^1
10^0
10^{-1}
16
18
20
22
24
26
28
30
number of vertices v

Teoretické a empirické obmedzenia ($k = 4$)

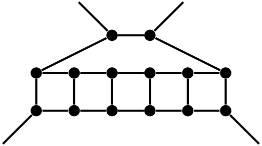




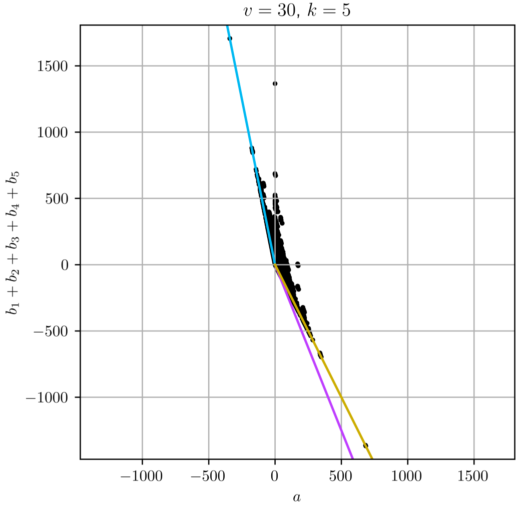
$a \geq 0$



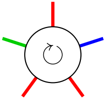
$a + b_1 + b_2 \geq 0$



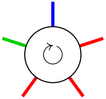
Teoretické a empirické obmedzenia ($k = 5$)



$$b = b_1 + b_2 + b_3 + b_4 + b_5$$

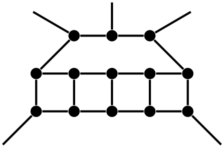


$5a + b \geq 0$



$5a + 2b \geq 0$

$2a + b \overset{?}{\geq} 0$



Teoretické a empirické obmedzenia ($k = 5$)

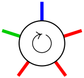
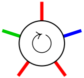
$$5a + 2b = \frac{1}{3}(5a + b) + \underbrace{\frac{5}{3}(2a + b)}_{\overset{?}{\geq} 0}$$

$$\frac{5a + 2b}{5a + b} \overset{?}{\geq} \frac{1}{3}$$

Teoretické a empirické obmedzenia ($k = 5$)

$$5a + 2b = \frac{1}{3}(5a + b) + \underbrace{\frac{5}{3}(2a + b)}_{\overset{?}{\geq} 0}$$

$$\frac{5a + 2b}{5a + b} \overset{?}{\geq} \frac{1}{3}$$

 :  $\overset{?}{\geq} 1 : 3$

Hypotéza

Pre každý ofarbiteľný planárny kubický 5-pól tvoria farbenia typu 00012 aspoň 1/4 celkového počtu farbení.

Teoretické a empirické obmedzenia ($k = 5$)

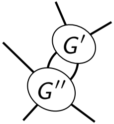
Hypotéza

Pre každý ofarbiteľný planárny kubický 5-pól tvoria farbenia typu 00012 aspoň 1/4 celkového počtu farbení.

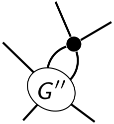
Veta

Najmenší protipríklad (ak existuje):

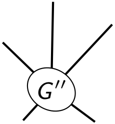
- je súvislý,
- neobsahuje most,
- neobsahuje 2-rez *okrem dvojíc hrán susedných s trčiacou*,
- má aspoň 36 vrcholov.



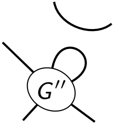
$= a' \cdot$

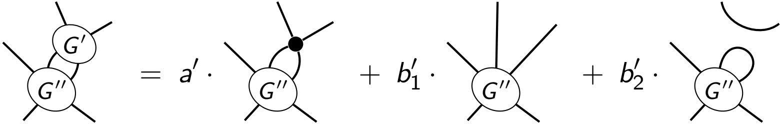


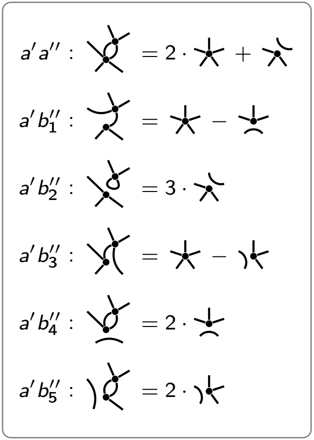
$+ \; b'_1 \cdot$

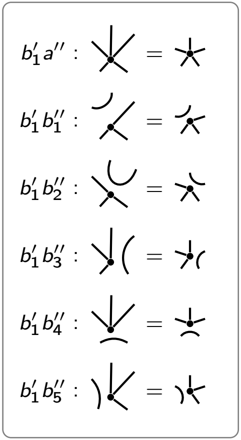


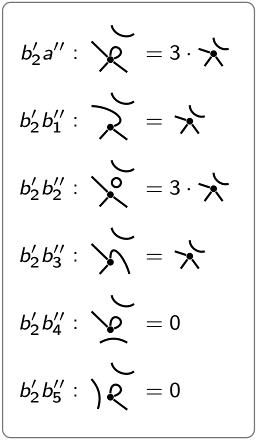
$+ \; b'_2 \cdot$




G'
G''
= a' ·
G''
+ b'_1 ·
G''
+ b'_2 ·
G''


a' a'' : = 2 · +
a' b''_1 : = −
a' b''_2 : = 3 ·
a' b''_3 : = −
a' b''_4 : = 2 ·
a' b''_5 : = 2 ·


b'_1 a'' : =
b'_1 b''_1 : =
b'_1 b''_2 : =
b'_1 b''_3 : =
b'_1 b''_4 : =
b'_1 b''_5 : =


b'_2 a'' : = 3 ·
b'_2 b''_1 : =
b'_2 b''_2 : = 3 ·
b'_2 b''_3 : =
b'_2 b''_4 : = 0
b'_2 b''_5 : = 0

$$2a + b_1 + b_2 + b_3 + b_4 + b_5$$

$$2a + b_1 + b_2 + b_3 + b_4 + b_5$$

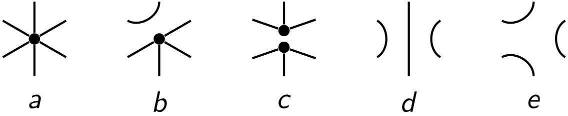
$$=$$

$$\begin{aligned} & a'\left(2(a'' + b_2'') + (a'' + b_4'' + b_5'')\right) \\ + & (a' + b_1')\left(2a'' + b_1'' + b_2'' + b_3'' + b_4'' + b_5''\right) \\ + & b_2'\left((a'' + b_1'' + b_2'') + (a'' + b_2'' + b_3'') + (a'' + b_2'')\right) \end{aligned}$$

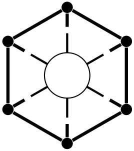
$$=$$

$$\begin{aligned} & (a' + b_1' + b_2')(2a'' + b_1'' + b_2'' + b_3'' + b_4'' + b_5'') \\ + & 2(a' + b_2')(a'' + b_2'') \\ + & (a' - b_2')(a'' + b_4'' + b_5'') \end{aligned}$$

Teoretické a empirické obmedzenia ($k = 6$)



$\circlearrowright\ \beta$	a	b	c	d	e	constraint
111111	1	1	0	1	1	T1 = E1
111212	6	2	0	0	0	T2 = E2
112112	3	1	0	1	0	T3 = E3
112233	2	1	0	0	1	T4 = E4
112332	3	1	1	1	0	T5 = E5
121323	3	0	2	0	0	T6 = E6
123123	1	0	1	0	0	T7 = E7
111122	6	4	0	2	3	T8 = 2 E1 + E8
112323	6	1	2	0	0	T9 = $\frac{1}{2}$ E2 + E6
	4	2	0	0	1	E8
	4	2	1	1	1	E9
	9	3	2	1	3	E10



1/8 všetkých farbení

Záver

Výsledky:

- koeficienty ako nástroj na štúdium farbení
- indikátory (p, r)-súvislosti a (p, r)-2-súvislosti
- návrh a implementácia algoritmu na výpočet koeficientov
- výpočet koeficientov pre planárne kubické k-póly do $\sim$ 30 vr.
- analýza teoretických a empirických obmedzení pre koeficienty
- hypotéza o minimálnom pomere počtu farbení v 5-póloch

Ďalší výskum:

- dôkaz hypotézy?
- systematickejšia analýza pre obmedzenia dané dokresleniami na planárny snark