Homework Set #1

1. Let $(S, +, \cdot, 0, 1)$ be a semiring. A *subsemiring* of S is a subset T of S, containing both 0 and 1 and closed under $+$ and $\cdot$, equipped by the restrictions of $+$ and $\cdot$ to $T \times T$. Suppose that S is a complete semiring. Given a subsemiring T of S, can one always find a suitable restriction of the infinite sum operator under which T is complete as well?

2. A *formal Laurent series* in variable z with coefficients in $\mathbb{C}$ takes the form
$$r = \sum_{n=N}^{\infty} a_n z^n,$$
for some (possibly negative) integer N. Define these objects formally, similarly to the definition of formal power series. Show that the set $\mathbb{C}(\!(z)\!)$ of all such series forms a field together with natural extensions of the sum and Cauchy product on $\mathbb{C}[\![z]\!]$ to $\mathbb{C}(\!(z)\!)$. If familiar with the concept, prove that $\mathbb{C}(\!(z)\!)$ is a fraction field of $\mathbb{C}[\![z]\!]$.

3. Let S be a semiring, Σ an alphabet, and $r, s \in S\langle\!\langle \Sigma^* \rangle\!\rangle$. The *Hadamard product* of r and s is a series $r \odot s \in S\langle\!\langle \Sigma^* \rangle\!\rangle$ defined by $(r \odot s, w) = (r, w)(s, w)$ for all $w \in \Sigma^*$.
   a) Show that the Hadamard product of series from $\mathbb{B}\langle\!\langle \Sigma^* \rangle\!\rangle$ can be interpreted as the usual intersection operation on formal languages.
   b) Show that given a semiring S and alphabet Σ, the set of all series in $S\langle\!\langle \Sigma^* \rangle\!\rangle$ rational over S might not be closed under the Hadamard product.
   c) Prove that the set of all series in $S\langle\!\langle \Sigma^* \rangle\!\rangle$ rational over S is closed under Hadamard product whenever the semiring S is commutative.

4. Let $r \in \mathbb{N}\langle\!\langle \Sigma^* \rangle\!\rangle$ be a series rational over $\mathbb{N}$. Prove that $\mathrm{supp}(r)$ is a rational (regular) language.

5. Let $r \in \mathbb{Z}\langle\!\langle \Sigma^* \rangle\!\rangle$ be a series rational over $\mathbb{Z}$. Show that the language $\mathrm{supp}(r)$ might not be rational (regular).

6. Find a series $r \in \mathbb{N}\langle\!\langle \Sigma^* \rangle\!\rangle$ rational over $\mathbb{Z}$ that is not rational over $\mathbb{N}$.

7. A weighted automaton (without ε-transitions) is *deterministic* if it has at most one state with nonzero initial weight and at most one transition upon each letter leading from each state. Show that weighted automata are not determinisable in general. In particular, show that there is a series $r \in \mathbb{N}\langle\!\langle \Sigma^* \rangle\!\rangle$ rational over $\mathbb{N}$ such that r is not realised by any deterministic weighted automaton over $\mathbb{N}$ and Σ.

8. Show that weighted automata over finite semirings are always determinisable.

9. An instance of the *Post's correspondence problem* (PCP) over a binary alphabet is a triple (Σ, f, g), where Σ is an alphabet and $f, g \colon \Sigma^* \to \{0,1\}^*$ are homomorphisms. The task is to decide whether there exists a nonempty word $w \in \Sigma^+$ such that $f(w) = g(w)$.[1]
   a) Given a PCP instance (Σ, f, g), describe a weighted automaton $\mathcal{A}$ over $\mathbb{Z}$ and Σ such that $(\|\mathcal{A}\|, w) = 0$ if and only if $f(w) = g(w)$.
   b) Deduce that the following problem is undecidable: given a series $r \in \mathbb{Z}\langle\!\langle \Sigma^* \rangle\!\rangle$ rational over $\mathbb{Z}$ (described by a weighted automaton), decide whether $(r, w) = 0$ for some $w \in \Sigma^*$.

10. Consider the following decision problem: given a series $r \in \mathbb{Q}\langle\!\langle \Sigma^* \rangle\!\rangle$ rational over $\mathbb{Q}$ (described by a weighted automaton), decide whether
$$r = \sum_{w \in \Sigma^*} qw$$
for some $q \in \mathbb{Q}$. Is this problem decidable?

[1] In the popular interpretation via „dominoes“, Σ can be thought of as the set of all tiles, while the homomorphisms map each tile to the words that it contains.

11. Consider the following decision problem: given a series $r \in \mathbb{Q}\langle\langle\Sigma^*\rangle\rangle$ rational over $\mathbb{Q}$ (described by a weighted automaton), decide whether $(r, w) \in \{0, 1\}$ for all $w \in \Sigma^*$. Is this problem decidable? (*Hint*: use closure of $\mathbb{Q}-\mathrm{Rat}(\Sigma^*)$ under Hadamard product, as described in Exercise 3.)

12. Consider the following decision problem: given a weighted automaton $\mathcal{A}$ over the tropical semiring $(\mathbb{N} \cup \{\infty\}, \min, +, \infty, 0)$ and $k \in \mathbb{N}$, decide whether there exists a word $w \in \Sigma^*$ such that $(\|\mathcal{A}\|, w) \leq k$. Is this problem decidable?