Homework Set #2

1. Find all integers $n \geq 2$, for which there exists a *real* $n \times n$ matrix A such that its spectrum $\sigma(A)$ satisfies $\sigma(A) \subseteq \mathbb{C} \setminus \mathbb{R}$.

2. Let $p(z) = z^k + d_{k-1}z^{k-1} + \ldots + d_0 z^0$ with $d_0, \ldots, d_{k-1} \in \mathbb{C}$ be a monic polynomial with complex coefficients and A_p the corresponding companion matrix.[1]

   a) Find the geometric multiplicities of the eigenvalues of A_p.
   b) Find a necessary and sufficient condition on p under which A_p is diagonalisable.
   c) Find a necessary and sufficient condition on p under which A_p is of full rank.

3. Given $n \in \mathbb{N}$, let $\mathcal{I}_n$ denote the set of all (nonnegative real) irreducible $n \times n$ matrices. Is $\mathcal{I}_n$ closed under:

   a) sum;
   b) product;
   c) nonnegative integer powers;
   d) positive integer powers?

4. Given $n \in \mathbb{N}$, let $\mathcal{P}_n$ denote the set of all (nonnegative real and irreducible) primitive $n \times n$ matrices. Is $\mathcal{P}_n$ closed under:

   a) sum;
   b) product;
   c) nonnegative integer powers;
   d) positive integer powers?

5. Let A be an $n \times n$ nonnegative real matrix and suppose that A^t is

   a) irreducible;
   b) primitive

   for some integer $t > 0$. Does A necessarily have to exhibit the same property?

6. Let $n \geq 2$ and A a nonnegative real $n \times n$ matrix that is both irreducible and symmetric (e.g., an adjacency matrix of a connected undirected graph). Determine all possible values of the imprimitivity index[2] of A.

7. Show that a sequence of complex numbers $(a_t)_{t=0}^{\infty}$ satisfies a recurrence

   $$a_{t+k} = c_{k-1}a_{t+k-1} + \ldots + c_0 a_t \qquad \text{for all } t \in \mathbb{N}$$

   for some $k \in \mathbb{N} \setminus \{0\}$ and $c_0, \ldots, c_{k-1} \in \mathbb{C}$ if and only if the corresponding generating function is rational.

8. Relate the ordinary and Laplacian spectra of *regular* graphs.

9. Show that for all $m, n \in \mathbb{N} \setminus \{0\}$, the number of spanning trees of the complete bipartite graph $K_{m,n}$ equals $m^{n-1}n^{m-1}$.

[1] Sprievodná matica.
[2] Index primitívnosti.

Let $\mathcal{G}$ be an undirected graph (simple, without loops) with vertex set $[n]$. A *signless Laplacian matrix* $Q(\mathcal{G})$ of $\mathcal{G}$ is defined by

$$Q(\mathcal{G}) = D + \mathrm{Adj}(\mathcal{G}),$$

where $\mathrm{Adj}(\mathcal{G})$ is the adjacency matrix of the graph $\mathcal{G}$ and $D = (d_{i,j})_{n\times n}$ is a diagonal matrix such that $d_{k,k} = \deg_{\mathcal{G}}(k)$ for $k = 1, \ldots, n$.

10. Show that the signless Laplacian matrix of a graph $\mathcal{G}$ with vertex set $[n]$ is always positive semidefinite and find a suitable $m \times n$ matrix B such that $Q(\mathcal{G}) = B^T B$.

11. All eigenvalues of a positive semidefinite matrix $Q(\mathcal{G})$ have to be nonnegative. Describe the class of all graphs $\mathcal{G}$ for which zero is an eigenvalue of $Q(\mathcal{G})$.

12. Show that the characteristic polynomials of the Laplacian and the signless Laplacian matrix coincide for bipartite graphs.