Finitely Ambiguous and Finitely Sequential Weighted Automata over Fields*

Peter Kostolányi[1]

Department of Computer Science, Comenius University in Bratislava, Mlynská dolina, 842 48 Bratislava, Slovakia

Abstract

Expressive power of finitely ambiguous and finitely sequential weighted automata over fields is examined. Rational series realised by such automata can be classified into infinite hierarchies according to ambiguity and sequentiality degrees of realising automata – it is shown that both these hierarchies are strict already over unary alphabets in case the field under consideration is not locally finite. Moreover, the expressive power of finitely sequential, finitely ambiguous, and polynomially ambiguous weighted automata is compared over different fields, both for unary and arbitrary finite alphabets, drawing a complete picture of the relations between corresponding classes of formal power series.

Keywords: Weighted automaton, Degree of ambiguity, Degree of sequentiality, Field, Hierarchy

1. Introduction

This article aims to contribute to a line of research studying *weighted automata of restricted ambiguity*, especially in terms of their expressive power. Notions like *unambiguity*, *finite ambiguity*, and *polynomial ambiguity* recently received considerable attention in the context of weighted automata, motivation for their study coming from several different directions. For instance, classes of weighted automata of restricted ambiguity were often considered in connection to various decision problems for weighted automata – several problems undecidable or of unknown decidability status for general weighted automata have been shown to be decidable when one restricts their scope to finitely or polynomially ambiguous weighted automata. Such results are known, e.g., for determinisation of weighted automata over tropical semirings [23, 24, 25], as well as in the setting of probabilistic automata [7, 11, 18]. Moreover, restricted ambiguity in weighted automata over the rational numbers and unary alphabets was considered with motivation coming from the study of decision problems for linear recurrences such as the Skolem problem [3, 4], and various classes of weighted automata with restricted ambiguity also arise in connection with the weighted first-order logic of M. Droste and P. Gastin [12, 33]. Furthermore, J. Bell and D. Smertnig [5] relatively recently proved that unambiguous weighted automata over fields realise precisely the class of noncommutative rational Pólya series, settling a long-standing conjecture of C. Reutenauer.

Although various observations about the *expressive power* of weighted automata with restricted ambiguity have been recorded for a relatively long time, it was only recently that this research crystallised into a more systematic study of so-called *ambiguity hierarchies* [3, 4, 9]. The ambiguity hierarchy over a semiring S consists of the classes of series realised by the unambiguous, finitely ambiguous, polynomially ambiguous, and unrestricted weighted automata over S; sometimes also the class of series realised by the deterministic weighted automata is added as its lowermost layer. This hierarchy was observed to be strict

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Email address: kostolanyi@fmph.uniba.sk (Peter Kostolányi)
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over tropical semirings by A. Chattopadhyay et al. [9]. Over the rational numbers, its strictness was established already for unary alphabets by C. Barloy et al. [3, 4]; some of their results also follow, to some extent, from the findings of [31, 12]. It is also observed in [3, 4] that the infinite hierarchy of series realised by k-ambiguous unary weighted automata over the rationals, for $k = 0, 1, 2, \ldots$, is strict as well.

Another concept somewhat related to finite ambiguity and considered in this article is that of *finite sequentiality* [2], also known as *multisequentiality* [10]. A weighted automaton is finitely sequential if it can be described as a deterministic weighted automaton with possibly more than one initial state. Such an automaton is thus always equivalent to a finite union of deterministic weighted automata. Every finitely sequential automaton is at the same time finitely ambiguous, but a finitely ambiguous weighted automaton might not even admit a finitely sequential equivalent [2].

The above-mentioned restrictions were studied not only for weighted finite automata over words, but also for weighted tree automata [30, 34, 35, 36, 37], and ambiguity hierarchies were also considered in the setting of weighted context-free grammars [20].

In this article, we study restricted ambiguity in automata with weights taken from an abstract *field*. Weighted automata over fields are known for their particularly rich and well-developed theory and abundance of appealing properties [8, 38]. Although the history of research on weighted automata over fields goes back to the foundational article of M.-P. Schützenberger [41], it still represents an active area [5, 6, 22].

It is thus quite surprising that very little was known about ambiguity hierarchies of rational series over fields, where the research was until recently limited to the particular case of automata over the rational numbers [3, 4]; to the author's knowledge, nothing at all was known about finitely sequential automata neither over fields in general, nor over particular fields such as the rationals. Recently, the relations between the expressive power of *polynomially ambiguous* and *unrestricted* weighted automata over fields were examined by the author [27]. In particular, it was shown [27] that unrestricted weighted automata over fields of characteristic zero that are not algebraically closed are more powerful than polynomially ambiguous weighted automata over the same field – already over unary alphabets. On the contrary, *unary* weighted automata over algebraically closed fields always admit polynomially ambiguous equivalents, regardless of the field's characteristic [27].

This article essentially continues in the above-mentioned research, the questions considered here being in a sense complementary to those studied in [27]. In particular, we mostly deal here with *finitely ambiguous* weighted automata over fields, for which we compare their expressive power with *polynomially ambiguous* automata. We also initiate the study of *finitely sequential* weighted automata over fields, and compare their expressive power with *finitely ambiguous* automata. Such comparisons of expressive power are done both over unary and over arbitrary finite alphabets, and mostly amount to uncovering the conditions on the underlying field, under which the inclusions between the corresponding classes of series are strict. All these questions are answered completely in this article.

Moreover, we study the *finite ambiguity hierarchy* composed by classes of series realised by k*-ambiguous* weighted automata for $k = 0, 1, 2, \ldots$, as well as the *finite sequentiality hierarchy*, which consists of classes of series realised by k*-sequential* weighted automata for $k = 0, 1, 2, \ldots$, and fully explore the conditions for their strictness over fields.

More precisely, when it comes to the relations between *finitely sequential* and *finitely ambiguous* weighted automata over fields, we show that all finitely ambiguous weighted automata over unary alphabets do actually admit finitely sequential equivalents – in fact, this result also holds for unary weighted automata over an arbitrary *commutative semiring*; the degree of sequentiality of an equivalent automaton is linked to a structural measure of the original finitely ambiguous automaton. On the other hand, we prove that finitely ambiguous weighted automata over fields and alphabets with at least two letters are strictly stronger than their finitely sequential counterparts whenever the underlying field is not locally finite. As the corresponding classes of series trivially coincide over locally finite fields, the relations between the power of finitely sequential and finitely ambiguous weighted automata over fields are fully characterised by these observations.

We also compare the expressive power of *finitely ambiguous* and *polynomially ambiguous* weighted automata over fields. Over unary alphabets, we show that polynomially ambiguous automata are strictly stronger if and only if the underlying field is of characteristic zero. On the other hand, the inclusion is strict over larger alphabets if and only if the underlying field is not locally finite. Note that no field of characteristic zero is locally finite, but a field that is not locally finite might not be of characteristic zero. Again, the relations between finitely and polynomially ambiguous weighted automata are fully explored when it comes to their expressive power over fields.

Finally, we study the *finite ambiguity hierarchy* and the *finite sequentiality hierarchy*, and prove that both are strict if and only if the underlying field is not locally finite – in fact, unary alphabets are sufficient to establish this observation. The relations between the classes of series realised by k-ambiguous and k-sequential weighted automata over fields for different $k \in \mathbb{N}$ are thus understood as well.

Part of the research presented herein already appeared in the conference article [26]. Newly added material includes the results dealing specifically with larger than unary alphabets – especially Theorem 6.1 and Theorem 6.4 – and a stronger statement of Theorem 5.8.

2. Preliminaries

Alphabets are understood to be finite and nonempty. We denote by $\mathbb{N}$ the set of all *nonnegative* integers and write $[n] = \{1, \ldots, n\}$ for each $n \in \mathbb{N}$.

Some familiarity with basic concepts of abstract algebra is assumed at the part of the reader, especially when it comes to topics related to fields and polynomials over them; see, e.g., [15, 19] for a reference. Fields are always understood to be commutative in what follows. Recall that a *field* is called *locally finite* when its finitely generated subfields are all finite; a *semiring* is *locally finite* when its finitely generated subsemirings are all finite. A field is locally finite if and only if it is locally finite as a semiring. The set of all $m \times n$ matrices over a set S is denoted by $S^{m\times n}$. The set $S^{n\times n}$ of all $n \times n$ matrices over a *semiring* S again forms a semiring together with the usual matrix addition and multiplication, the $n \times n$ zero matrix $\mathbf{0}_n$ as the additive neutral element, and the $n \times n$ identity matrix $\mathbf{I}_n$ as the multiplicative neutral element.

2.1. Formal Power Series and Weighted Automata

Consult, e.g., [8, 13, 14, 38, 40] for a comprehensive reference on weighted automata and formal power series in several noncommutative variables. We now briefly recall some fundamental concepts in this area that are used in this article.

A (noncommutative) *formal power series* over a semiring S and alphabet Σ is a mapping $r\colon \Sigma^* \to S$ interpreted as follows: the value of r upon $w \in \Sigma^*$ is denoted by (r, w) and called the *coefficient* of w in r; we then write

$$r = \sum_{w\in\Sigma^*} (r, w)\, w.$$

The set of all formal power series over S and Σ is denoted by $S\langle\langle\Sigma^*\rangle\rangle$.

The algebra on $S\langle\langle\Sigma^*\rangle\rangle$ is given by the operations of *sum*, defined for all $r, s \in S\langle\langle\Sigma^*\rangle\rangle$ to be a series $r + s$ such that $(r + s, w) = (r, w) + (s, w)$ for all $w \in \Sigma^*$, and *Cauchy product*, defined for all $r, s \in S\langle\langle\Sigma^*\rangle\rangle$ to be a series $r \cdot s$ such that for each $w \in \Sigma^*$,

$$(r \cdot s, w) = \sum_{\substack{u,v\in\Sigma^*\\ uv=w}} (r, u)(s, v).$$

Note that this definition of a multiplicative operation is actually the reason behind viewing the mappings from Σ^* to S as noncommutative formal power series.

Each $a \in S$ is identified with the series $r_a \in S\langle\langle\Sigma^*\rangle\rangle$ such that $(r_a, \varepsilon) = a$ and $(r_a, w) = 0$ for all $w \in \Sigma^+$, and each $w \in \Sigma^*$ with the series $r_w \in S\langle\langle\Sigma^*\rangle\rangle$ such that $(r_w, w) = 1$ and $(r_w, x) = 0$ for all $x \in \Sigma^* \setminus \{w\}$. Given a semiring S and alphabet Σ, the algebra $(S\langle\langle\Sigma^*\rangle\rangle, +, \cdot, 0, 1)$ is a semiring as well.

A family $(r_i \mid i \in I)$ of series from $S\langle\langle\Sigma^*\rangle\rangle$, where I is some index set, is said to be *locally finite* if the set $I(w) = \{i \in I \mid (r_i, w) \neq 0\}$ is finite for all $w \in \Sigma^*$. One can then define the *sum* over this family by

$$\sum_{i\in I} r_i = r,$$

where $r \in S\langle\langle\Sigma^*\rangle\rangle$ is a series with (r, w) given, for each $w \in \Sigma^*$, by a *finite* sum

$$(r, w) = \sum_{i \in I(w)} (r_i, w).$$

A typical example of a locally finite family of series is given by $(r^t \mid t \in \mathbb{N})$, where $r \in S\langle\langle\Sigma^*\rangle\rangle$ is a *proper* series – *i.e.*, one has $(r, \varepsilon) = 0$. It is thus possible to define the *star* of a proper series $r \in S\langle\langle\Sigma^*\rangle\rangle$ by

$$r^* = \sum_{t \in \mathbb{N}} r^t.$$

A *weighted (finite) automaton* over a semiring S and over an alphabet Σ is a quadruple $\mathcal{A} = (Q, \sigma, \iota, \tau)$ with Q being a finite set of states, $\sigma \colon Q \times \Sigma \times Q \to S$ a transition weighting function, $\iota \colon Q \to S$ an initial weighting function, and $\tau \colon Q \to S$ a terminal weighting function.

A *transition* in the automaton $\mathcal{A}$ is a triple $(p, c, q) \in Q \times \Sigma \times Q$ such that $\sigma(p, c, q) \neq 0$. A *run* of the automaton $\mathcal{A}$ is a word $\gamma = q_0 c_1 q_1 c_2 q_2 \ldots q_{t-1} c_t q_t \in Q(\Sigma Q)^*$ with $t \in \mathbb{N}$, $q_0, \ldots, q_t \in Q$, and $c_1, \ldots, c_t \in \Sigma$ such that (q_{k-1}, c_k, q_k) is a transition in $\mathcal{A}$ for $k = 1, \ldots, t$. We also say that γ is a *run on the word* $c_1 \ldots c_t$ *leading from* q_0 *to* q_t, and call q_0 the *initial state* of γ and q_t the *terminal state* of γ. Moreover, we say that γ is *successful* if $\iota(q_0) \neq 0$ and $\tau(q_t) \neq 0$. The *label* of the run γ is the word

$$\lambda(\gamma) = c_1 \ldots c_t,$$

the *pure value* of γ is the element of S given by

$$\sigma(\gamma) = \sigma(q_0, c_1, q_1)\sigma(q_1, c_2, q_2) \ldots \sigma(q_{t-1}, c_t, q_t),$$

and the *complete value* of γ is defined by

$$\overline{\sigma}(\gamma) = \iota(q_0)\sigma(\gamma)\tau(q_t).$$

The number $|\gamma| = t$ is called the *length* of γ. The set of all runs of $\mathcal{A}$ on w is denoted by $\mathcal{R}(\mathcal{A}, w)$ and the set of all successful runs of $\mathcal{A}$ on w by $\mathcal{R}_s(\mathcal{A}, w)$. We then also write

$$\mathcal{R}(\mathcal{A}) = \bigcup_{w \in \Sigma^*} \mathcal{R}(\mathcal{A}, w) \qquad \text{and} \qquad \mathcal{R}_s(\mathcal{A}) = \bigcup_{w \in \Sigma^*} \mathcal{R}_s(\mathcal{A}, w).$$

The *behaviour* of a weighted automaton $\mathcal{A} = (Q, \sigma, \iota, \tau)$ over a semiring S and alphabet Σ is a formal power series $\|\mathcal{A}\| \in S\langle\langle\Sigma^*\rangle\rangle$ given by

$$(\|\mathcal{A}\|, w) = \sum_{\gamma \in \mathcal{R}_s(\mathcal{A}, w)} \overline{\sigma}(\gamma) = \sum_{\gamma \in \mathcal{R}(\mathcal{A}, w)} \overline{\sigma}(\gamma)$$

for all $w \in \Sigma^*$, both sums being obviously finite. This is equivalent to the observation that the families of series $(\overline{\sigma}(\gamma)\,\lambda(\gamma) \mid \gamma \in \mathcal{R}_s(\mathcal{A}))$ and $(\overline{\sigma}(\gamma)\,\lambda(\gamma) \mid \gamma \in \mathcal{R}(\mathcal{A}))$ are locally finite, implying that the behaviour of $\mathcal{A}$ can also be defined by

$$\|\mathcal{A}\| = \sum_{\gamma \in \mathcal{R}_s(\mathcal{A})} \overline{\sigma}(\gamma)\,\lambda(\gamma) = \sum_{\gamma \in \mathcal{R}(\mathcal{A})} \overline{\sigma}(\gamma)\,\lambda(\gamma).$$

We then say that the series $\|\mathcal{A}\|$ is *realised* by $\mathcal{A}$. A series $r \in S\langle\langle\Sigma^*\rangle\rangle$ is *rational*[2] over S if it is realised by a weighted finite automaton over S and Σ.

[2] This terminology stems from the fact that weighted automata are equivalent to weighted rational – or regular – expressions, which can be seen as a common generalisation of rational (regular) expressions without weights and of rational fractions over the complex or real numbers. This connection to rational fractions also implies that the class of series realised by weighted automata forms a generalisation of the class of all formal Maclaurin expansions of rational functions analytic at the origin of the complex plane, which is obtained when taking the field $\mathbb{C}$ of complex numbers for S and a unary alphabet $\Sigma = \{z\}$. Note that some authors use a different nomenclature here, strictly reserving the attribute "rational" to the context of rational expressions [14, 17]. Our terminology follows J. Sakarovitch [38, 39].

A weighted automaton $\mathcal{A} = (Q, \sigma, \iota, \tau)$ over S and Σ is *accessible* if for each $q \in Q$, there exists a run of $\mathcal{A}$ from some p with $\iota(p) \neq 0$ to q; *coaccessible* if for each $p \in Q$, there exists a run of $\mathcal{A}$ from p to some q with $\tau(q) \neq 0$; and *trim* if it is both accessible and coaccessible.

In what follows, we often without loss of generality confine ourselves to automata with state sets $[n]$ for $n \in \mathbb{N}$ – we then write $\mathcal{A} = (n, \sigma, \iota, \tau)$ instead of $\mathcal{A} = ([n], \sigma, \iota, \tau)$. Moreover, we apply the standard graph-theoretic terminology to weighted automata. This refers to a directed multigraph whose vertices are states of the automaton, while for each pair of states p, q, the transitions of the form (p, c, q) correspond bijectively to directed edges from p to q.

2.2. Finite Sequentiality and Restricted Ambiguity in Weighted Automata

A weighted automaton $\mathcal{A} = (Q, \sigma, \iota, \tau)$ over S and Σ is said to be *k-sequential* for $k \in \mathbb{N}$ if there are at most k distinct states $q \in Q$ satisfying $\iota(q) \neq 0$, and if $\sigma(p, c, q) \neq 0$ with $\sigma(p, c, q') \neq 0$ imply $q = q'$ for all $p, q, q' \in Q$ and all $c \in \Sigma$. In particular, 1-sequential automata are typically termed *deterministic* or *sequential* [29].[3] A weighted automaton $\mathcal{A}$ is *finitely sequential* [2] if it is k-sequential for some $k \in \mathbb{N}$.[4] A series is realised by a k-sequential automaton if and only if it can be expressed as a sum of k series realised by deterministic weighted automata.

The *ambiguity degree* of a weighted automaton $\mathcal{A}$ is given by a function $\mathrm{amb}_{\mathcal{A}} \colon \Sigma^* \to \mathbb{N}$ counting successful runs of $\mathcal{A}$ on words over Σ; that is, $\mathrm{amb}_{\mathcal{A}}(w) = |\mathcal{R}_s(\mathcal{A}, w)|$ for all $w \in \Sigma^*$. The automaton $\mathcal{A}$ is said to be *k-ambiguous* for $k \in \mathbb{N}$ if $\mathrm{amb}_{\mathcal{A}}(w) \leq k$ for all $w \in \Sigma^*$, while 1-ambiguous automata are called *unambiguous.* An automaton $\mathcal{A}$ is *finitely ambiguous* if it is k-ambiguous for some $k \in \mathbb{N}$ and *polynomially ambiguous* if there exists a polynomial function $p \colon \mathbb{N} \to \mathbb{N}$ such that $\mathrm{amb}_{\mathcal{A}}(w) \leq p(|w|)$ for all $w \in \Sigma^*$.

Given a semiring S and alphabet Σ, we denote:

- By $\mathsf{Det}(S, \Sigma)$ the set of all series realised by *deterministic* weighted automata over S and Σ;
- By $\mathsf{k\text{-}Seq}(S, \Sigma)$, for $k \in \mathbb{N}$, the set of all series realised by *k-sequential* automata over S and Σ;
- By $\mathsf{FinSeq}(S, \Sigma)$ the set of all series realised by *finitely sequential* automata over S and Σ;
- By $\mathsf{UnAmb}(S, \Sigma)$, the set of all series realised by *unambiguous* automata over S and Σ;
- By $\mathsf{k\text{-}Amb}(S, \Sigma)$, for $k \in \mathbb{N}$, the set of all series realised by *k-ambiguous* automata over S and Σ;
- By $\mathsf{FinAmb}(S, \Sigma)$ the set of all series realised by *finitely ambiguous* automata over S and Σ;
- By $\mathsf{PolyAmb}(S, \Sigma)$ the set of all series realised by *polynomially ambiguous* automata over S and Σ;
- By $\mathsf{Rat}(S, \Sigma)$ the set of *all* series in $S\langle\langle \Sigma^* \rangle\rangle$ rational over S.

Every k-sequential weighted automaton, for $k \in \mathbb{N}$, is at the same time k-ambiguous, but the converse does not necessarily hold. We record this trivial observation as a proposition for later reference.

Proposition 2.1. *Let S be a semiring, Σ an alphabet, and $k \in \mathbb{N}$. Then every k-sequential weighted automaton over S and Σ is k-ambiguous. Hence* $\mathsf{k\text{-}Seq}(S, \Sigma) \subseteq \mathsf{k\text{-}Amb}(S, \Sigma)$.

The sets of series introduced above can obviously be classified into several hierarchies, not strict in general, summarised by the following proposition. The hierarchy (i) is often called the *ambiguity hierarchy* [3, 4, 30]. In what follows, we also refer to (iii) as to the *finite sequentiality hierarchy* and to (iv) as to the *finite ambiguity hierarchy.*

[3]Some authors also call such automata *subsequential*, while they reserve the term *sequential* for a more restricted class of automata. See S. Lombardy and J. Sakarovitch [29] for more information.

[4]Note that C. Allauzen and M. Mohri [1] use the term *finitely subsequential transducer* in a completely different sense.

Proposition 2.2. *Let S be a semiring and Σ an alphabet. Then:*

(i) $\mathsf{Det}(S,\Sigma) \subseteq \mathsf{UnAmb}(S,\Sigma) \subseteq \mathsf{FinAmb}(S,\Sigma) \subseteq \mathsf{PolyAmb}(S,\Sigma) \subseteq \mathsf{Rat}(S,\Sigma)$;

(ii) $\mathsf{Det}(S,\Sigma) \subseteq \mathsf{FinSeq}(S,\Sigma) \subseteq \mathsf{FinAmb}(S,\Sigma)$;

(iii) $\mathsf{k}\text{-}\mathsf{Seq}(S,\Sigma) \subseteq (\mathsf{k}+1)\text{-}\mathsf{Seq}(S,\Sigma)$ *for all* $k \in \mathbb{N}$*, while* $\mathsf{Det}(S,\Sigma) = 1\text{-}\mathsf{Seq}(S,\Sigma)$ *and*

$$\mathsf{FinSeq}(S,\Sigma) = \bigcup_{k=0}^{\infty} \mathsf{k}\text{-}\mathsf{Seq}(S,\Sigma);$$

(iv) $\mathsf{k}\text{-}\mathsf{Amb}(S,\Sigma) \subseteq (\mathsf{k}+1)\text{-}\mathsf{Amb}(S,\Sigma)$ *for all* $k \in \mathbb{N}$*, while* $\mathsf{UnAmb}(S,\Sigma) = 1\text{-}\mathsf{Amb}(S,\Sigma)$ *and*

$$\mathsf{FinAmb}(S,\Sigma) = \bigcup_{k=0}^{\infty} \mathsf{k}\text{-}\mathsf{Amb}(S,\Sigma).$$

We focus on the case when S is a field in this article, and we study the conditions, under which specific parts of the above-described hierarchies become strict depending on the underlying field and alphabet: we examine strictness of the inclusions $\mathsf{FinSeq}(S,\Sigma) \subseteq \mathsf{FinAmb}(S,\Sigma)$ and $\mathsf{FinAmb}(S,\Sigma) \subseteq \mathsf{PolyAmb}(S,\Sigma)$, as well as of the hierarchies (*iii*) and (*iv*). As a starting point, let us recall the following well-known observation (see, e.g., [29, Proposition 11]) implying that *almost all* inclusions of Proposition 2.2 actually become equalities when S is a locally finite semiring – or, in particular, a locally finite field.

Proposition 2.3. *Let S be a locally finite semiring. Then* $\mathsf{Det}(S,\Sigma) = \mathsf{Rat}(S,\Sigma)$.

2.3. Linear Representations

Weighted automata over a semiring S and alphabet Σ can also be viewed as *linear S-representations* over Σ, which are quadruples $\mathcal{P} = (n, \mathbf{i}, \mu, \mathbf{f})$ such that: $n \in \mathbb{N}$ is a dimension of the linear representation; $\mathbf{i} \in S^{1\times n}$ is a vector of initial weights; $\mu\colon (\Sigma^*, \cdot, \varepsilon) \to (S^{n\times n}, \cdot, \mathbf{I}_n)$ is a monoid homomorphism; and $\mathbf{f} \in S^{n\times 1}$ is a vector of terminal weights. The *series* $\|\mathcal{P}\|$ *realised by* $\mathcal{P}$ is given by

$$(\|\mathcal{P}\|, w) = \mathbf{i}\mu(w)\mathbf{f}$$

for all $w \in \Sigma^*$. A series $r \in S\langle\langle\Sigma^*\rangle\rangle$ is *recognisable* over S if it is realised by a linear S-representation.

The classes of recognisable and rational series *over words* coincide by a well-known classical result [38]. In fact, every weighted automaton $\mathcal{A} = (n, \sigma, \iota, \tau)$ over S and Σ corresponds to a linear S-representation $\mathcal{P}_{\mathcal{A}} = (n, \mathbf{i}, \mu, \mathbf{f})$, where $\mathbf{i} = (\iota(1), \ldots, \iota(n))$, the matrix $\mu(c) = (c_{i,j})_{n\times n}$ is given by $c_{i,j} = \sigma(i, c, j)$ for every $c \in \Sigma$ and $i, j = 1, \ldots, n$, and $\mathbf{f} = (\tau(1), \ldots, \tau(n))^T$. Clearly $\|\mathcal{P}_{\mathcal{A}}\| = \|\mathcal{A}\|$.

Consider in addition a mapping $\nu\colon S \to \mathbb{N}$ given for all $a \in S$ by

$$\nu(a) = \begin{cases} 1 & \text{if } a \neq 0, \\ 0 & \text{if } a = 0. \end{cases} \tag{1}$$

Applying this mapping componentwise to vectors and matrices, it is clear that

$$\mathrm{amb}_{\mathcal{A}}(c_1 \ldots c_t) = \nu(\mathbf{i})\nu(\mu(c_1)) \ldots \nu(\mu(c_t))\nu(\mathbf{f})$$

for all $t \in \mathbb{N}$ and $c_1, \ldots, c_t \in \Sigma$.

2.4. Rational Series over Unary Alphabets

We usually write a linear representation $\mathcal{P} = (n, \mathbf{i}, \mu, \mathbf{f})$ over a *unary alphabet* $\Sigma = \{c\}$ as $\mathcal{P} = (n, \mathbf{i}, A, \mathbf{f})$, where $A = \mu(c)$ is the only matrix needed to specify the homomorphism μ. This means that given a weighted automaton $\mathcal{A}$ over a semiring S and unary alphabet $\Sigma = \{c\}$ with $\mathcal{P}_{\mathcal{A}} = (n, \mathbf{i}, A, \mathbf{f})$,

$$\left(\|\mathcal{A}\|, c^t\right) = \mathbf{i}A^t\mathbf{f}$$

holds for all $t \in \mathbb{N}$. The automaton $\mathcal{A}$ can thus also be interpreted as an initial value problem for the system of linear difference equations (*i.e.*, recurrences)

$$\mathbf{x}_{t+1} = A\mathbf{x}_t \qquad \text{for all } t \in \mathbb{N},$$

the initial conditions being given by $\mathbf{x}_0 = \mathbf{f}$. When $S = \mathbb{F}$ is a *field*, the theory of difference equations [16] allows us to express the components of $\mathbf{x}_t$, and thus also $(\|\mathcal{A}\|, c^t)$, in closed form over the *algebraic closure* $\overline{\mathbb{F}}$ of $\mathbb{F}$. Indeed, by similarity of A to a matrix over $\overline{\mathbb{F}}$ in the Jordan canonical form, it follows[5] that for all $t \in \mathbb{N}$,

$$\left(\|\mathcal{A}\|, c^t\right) = \sum_{\lambda \in \mathrm{sp}(A)} \sum_{j=0}^{\alpha(\lambda)-1} a_{\lambda,j} \binom{t}{j} \lambda^{t-j}, \tag{2}$$

where $\mathrm{sp}(A)$ denotes the spectrum of A over $\overline{\mathbb{F}}$, the algebraic multiplicity of each eigenvalue λ of A is denoted by $\alpha(\lambda)$, and $a_{\lambda,j} \in \overline{\mathbb{F}}$ are constants for all $\lambda \in \mathrm{sp}(A)$ and $j = 0, \ldots, \alpha(\lambda) - 1$. Recall that the spectrum $\mathrm{sp}(A)$ contains precisely the roots over $\overline{\mathbb{F}}$ of the characteristic polynomial $\mathrm{ch}_A(x) = \det(x\mathbf{I}_n - A)$ of A, and that the algebraic multiplicity of $\lambda \in \mathrm{sp}(A)$ is its multiplicity as a root of $\mathrm{ch}_A(x)$.

The constants $a_{\lambda,j}$ of (2) are always *uniquely* determined as a solution to a linear system of equations given by (2) for $t = 0, \ldots, n-1$, in which they are the only unknowns. In particular, *every* choice of initial values on the left-hand sides uniquely determines the constants $a_{\lambda,j}$ and conversely, every choice of the constants $a_{\lambda,j}$ gives different initial values [16]. This observation can be established, e.g., as a consequence of the fact that the matrix of the above-mentioned linear system is the so-called Casorati matrix [16] of the functions $f_{\lambda,j}(t) = \binom{t}{j}\lambda^{t-j}$ for $\lambda \in \mathrm{sp}(A)$ and $j = 0, \ldots, \alpha(\lambda) - 1$. This is a generalised Vandermonde matrix [21, 16], so it is necessarily invertible. The linear system thus always has a unique solution. Moreover, the following observation important in its own right can be obtained as a consequence of invertibility of the above-described Casorati matrix for suitable $\mathrm{sp}(A)$ and $\alpha(\lambda)$ for all $\lambda \in \mathrm{sp}(A)$.

Theorem 2.4. *Let $\mathbb{F}$ be a field. Any finite set of pairwise distinct functions of the form $f_{\lambda,j}(t) = \binom{t}{j}\lambda^{t-j}$ with $\lambda \in \mathbb{F}$ and $j \in \mathbb{N}$ is linearly independent (over $\mathbb{F}$ or over its extension).*

Let us finally consider a weighted automaton $\mathcal{A}$, over any semiring S and over the unary alphabet $\Sigma = \{c\}$, such that $\mathcal{P}_{\mathcal{A}} = (n, \mathbf{i}, A, \mathbf{f})$. Let $\nu \colon S \to \mathbb{N}$ be given by (1). Then

$$\mathrm{amb}_{\mathcal{A}}(c^t) = \nu(\mathbf{i})\nu(A)^t\nu(\mathbf{f})$$

for all $t \in \mathbb{N}$, so that $\mathrm{amb}_{\mathcal{A}}(c^t)$ admits a closed form analogous to (2) over $\mathbb{C}$:

$$\mathrm{amb}_{\mathcal{A}}(c^t) = \sum_{\lambda \in \mathrm{sp}(\nu(A))} \sum_{j=0}^{\alpha'(\lambda)-1} a'_{\lambda,j} \binom{t}{j} \lambda^{t-j}, \tag{3}$$

where $\mathrm{sp}(\nu(A))$ denotes the spectrum of $\nu(A)$, the algebraic multiplicity of an eigenvalue λ of $\nu(A)$ is denoted by $\alpha'(\lambda)$, and $a'_{\lambda,j} \in \mathbb{C}$ for $\lambda \in \mathrm{sp}(\nu(A))$ and $j = 0, \ldots, \alpha'(\lambda) - 1$. We call $\nu(A)$ the *enumeration matrix* of $\mathcal{A}$ in what follows.

[5] Under the convention that $\binom{t}{j}0^{t-j} = 0$ for all natural numbers $t < j$; of course, $0^0 = 1$.

3. Characterisations of Finitely and Polynomially Ambiguous Automata

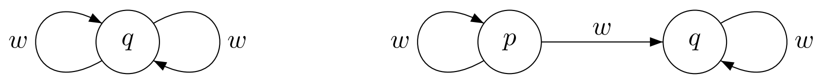
We now make some preliminary remarks on automata with restricted ambiguity. First, let us note that the ambiguity degree of a weighted automaton does not at all depend on its weights. This means that weights can be forgotten and the known criteria [42] for nondeterministic finite automata without weights can be applied to determine whether a given weighted automaton is, say, finitely or polynomially ambiguous. Let us recall these criteria, as described by A. Weber and H. Seidl [42].

Theorem 3.1 (A. Weber and H. Seidl [42])**.** *Let $\mathcal{A}$ be a trim finite automaton with state set Q over an alphabet Σ. Then $\mathcal{A}$ is:*

(i) Polynomially ambiguous *if and only if there does not exist a state q with at least two distinct runs from q to q upon some word $w \in \Sigma^*$;*

(ii) Finitely ambiguous *if and only if there is no pair of distinct states p, q such that for some $w \in \Sigma^*$, there are runs upon w from p to p, from p to q, as well as from q to q.*

The "forbidden configurations" for polynomial and finite ambiguity, described in Theorem 3.1, are schematically depicted in Fig. 1.



(a) For polynomial ambiguity. **(b)** For finite ambiguity.

Figure 1: The "forbidden configurations" for polynomially and finitely ambiguous trim finite automata, as identified by A. Weber and H. Seidl [42]. Distinct arrows represent distinct *runs*, as opposed to transitions.

The criteria of A. Weber and H. Seidl obviously admit a particularly simple form for unary automata, which we now make explicit.

Theorem 3.2. *Let $\mathcal{A}$ be a trim finite automaton over the unary alphabet $\Sigma = \{c\}$. Then $\mathcal{A}$ is:*

(i) Polynomially ambiguous *if and only if its strongly connected components are all either single vertices, or directed cycles;*

(ii) Finitely ambiguous *if and only if, in addition to (i), there is no run of $\mathcal{A}$ passing through two distinct directed cycles.*

As already mentioned, both Theorem 3.1 and Theorem 3.2 can also be applied to weighted automata over an arbitrary semiring.

The characterisations of Theorem 3.2 can also be obtained, for a unary weighted automaton $\mathcal{A}$ with $\mathcal{P}_\mathcal{A} = (n, \mathbf{i}, A, \mathbf{f})$, with a little help of Perron-Frobenius theory [32] applied to the enumeration matrix $\nu(A)$. Indeed, the condition (i) is equivalent to all eigenvalues of $\nu(A)$ being of absolute value 0 or 1. If this is the case, the expression on the right-hand side of (3) reduces to a polynomial function and $\mathcal{A}$ is polynomially ambiguous. Otherwise, the Perron-Frobenius theory gives us existence of an eigenvalue $\lambda > 1$ with at least one nonzero coefficient $a'_{\lambda,j}$ in (3) – the automaton $\mathcal{A}$ is not polynomially ambiguous. Given (i), the equivalence of (ii) with finite ambiguity can be easily established by noting that a possibility of passing through two different cycles in a single run is equivalent to unboundedness of $\mathrm{amb}_\mathcal{A}$.

The characterisation of polynomially ambiguous unary automata provided by Theorem 3.2 also has the following useful consequence.

Proposition 3.3. *Let $\mathbb{F}$ be a field and $\mathcal{A}$ a trim polynomially ambiguous unary weighted automaton over $\mathbb{F}$ and $\Sigma = \{c\}$ with $\mathcal{P}_\mathcal{A} = (n, \mathbf{i}, A, \mathbf{f})$. Then*

$$\mathrm{ch}_A(x) = x^{\ell_0} \prod_{k=1}^{s} \left(x^{\ell_k} - b_k\right)$$

for some $\ell_0, s \in \mathbb{N}$, $\ell_1, \ldots, \ell_s \in \mathbb{N} \setminus \{0\}$, and $b_1, \ldots, b_s \in \mathbb{F} \setminus \{0\}$.

Proof. The matrix A can be seen as the adjacency matrix of the automaton $\mathcal{A}$ viewed as a weighted directed graph. The characteristic polynomial of A can then be expressed as a product of characteristic polynomials of matrices corresponding to particular strongly connected components. By Theorem 3.2, each strongly connected component is either a single vertex without a loop – in which case its characteristic polynomial is x – or a directed cycle – in which case its characteristic polynomial is $x^\ell - b$, where ℓ is the length of the cycle, and b is the product of weights of all transitions forming the cycle. ☐

4. Some Closure Properties of $\mathsf{FinAmb}(S, \Sigma)$

Let us continue in our preliminary examinations by explicitly mentioning three simple closure properties of the sets of series realised by finitely ambiguous weighted automata. The observations made in this section are needed for the proof of Theorem 5.8 below.

Proposition 4.1. *Let $\mathbb{F}$ be a field, Σ an alphabet, $n \in \mathbb{N}$, $a_1, \ldots, a_n \in \mathbb{F}$, and $r_1, \ldots, r_n \in \mathsf{FinAmb}(\mathbb{F}, \Sigma)$. Let*

$$r = \sum_{i=1}^{n} a_i r_i.$$

Then r is in $\mathsf{FinAmb}(\mathbb{F}, \Sigma)$ as well.

Proof. Assume that for $i = 1, \ldots, n$, the series r_i is realised by a finitely ambiguous weighted automaton $\mathcal{A}_i = (Q_i, \sigma_i, \iota_i, \tau_i)$ over $\mathbb{F}$ and Σ; without loss of generality, let the state sets $Q_1, \ldots, Q_n$ be pairwise disjoint.

The series r is then clearly realised by a weighted automaton $\mathcal{A} = (Q, \sigma, \iota, \tau)$ over $\mathbb{F}$ and Σ, where:

$$Q = Q_1 \uplus \ldots \uplus Q_n;$$

$\sigma(p, c, q) = \sigma_i(p, c, q)$ for all $i \in [n]$, all $p, q \in Q_i$, and all $c \in \Sigma$; $\sigma(p, c, q) = 0$ for all $i, j \in [n]$ such that $i \neq j$, all $p \in Q_i$ and $q \in Q_j$, and all $c \in \Sigma$; $\iota(q) = a_i \iota_i(q)$ for all $i \in [n]$ and $q \in Q_i$; and $\tau(q) = \tau_i(q)$ for all $i \in [n]$ and all $q \in Q_i$.

Moreover, it is clear that for all $w \in \Sigma^*$,

$$\mathcal{R}_s(\mathcal{A}, w) \subseteq \mathcal{R}_s(\mathcal{A}_1, w) \uplus \ldots \uplus \mathcal{R}_s(\mathcal{A}_n, w),$$

so that

$$\mathrm{amb}_{\mathcal{A}}(w) \leq \sum_{i=1}^{n} \mathrm{amb}_{\mathcal{A}_i}(w).$$

This means that the automaton $\mathcal{A}$ is finitely ambiguous as well, and $r \in \mathsf{FinAmb}(\mathbb{F}, \Sigma)$. ☐

Proposition 4.2. *Let $\mathbb{F}$ be a field, Σ an alphabet, $r \in \mathsf{FinAmb}(\mathbb{F}, \Sigma)$, and $c \in \Sigma$. Then the series cr is in $\mathsf{FinAmb}(\mathbb{F}, \Sigma)$ as well.*

Proof. Let r be realised by a finitely ambiguous weighted automaton $\mathcal{A} = (n, \sigma, \iota, \tau)$ over $\mathbb{F}$ and Σ. Then cr is clearly realised by a weighted automaton $\mathcal{B} = (n+1, \sigma', \iota', \tau')$ over $\mathbb{F}$ and Σ such that: $\sigma'(p, a, q) = \sigma(p, a, q)$ for all $p, q \in [n]$ and $a \in \Sigma$; $\sigma'(n+1, c, q) = \iota(q)$ for all $q \in [n]$; $\sigma'(n+1, a, q) = 0$ for all $q \in [n]$ and $a \in \Sigma \setminus \{c\}$; $\sigma'(p, a, n+1) = 0$ for all $p \in [n+1]$ and $a \in \Sigma$; $\iota'(q) = 0$ for all $q \in [n]$; $\iota'(n+1) = 1$; $\tau'(q) = \tau(q)$ for all $q \in [n]$; and $\tau'(n+1) = 0$.

For all $w \in \Sigma^*$, there clearly is a bijection between $\mathcal{R}_s(\mathcal{A}, w)$ and $\mathcal{R}_s(\mathcal{B}, cw)$, which assigns to each $\gamma \in \mathcal{R}_s(\mathcal{A}, w)$ a run $\gamma' \in \mathcal{R}_s(\mathcal{B}, cw)$ that starts in the state $n+1$, takes a transition upon c to the initial state of γ, and subsequently continues in the same way as γ. In particular,

$$\mathrm{amb}_{\mathcal{B}}(cw) = \mathrm{amb}_{\mathcal{A}}(w)$$

for all $w \in \Sigma^*$, which, together with the obvious observation that

$$\mathrm{amb}_{\mathcal{B}}(x) = 0$$

for all $x \in \Sigma^* \setminus c\Sigma^*$, means that $\mathcal{B}$ is finitely ambiguous as well – thus $cr \in \mathsf{FinAmb}(\mathbb{F}, \Sigma)$. ☐

Proposition 4.3. *Let $\mathbb{F}$ be a field, $r \in \mathsf{FinAmb}(\mathbb{F}, \{c\})$, and $h\colon c^* \to c^*$ a homomorphism given by $h(c) = c^k$ for some $k \in \mathbb{N} \setminus \{0\}$. Then the series*

$$h(r) = \sum_{t \in \mathbb{N}} (r, c^t)\, h(c^t)$$

is in $\mathsf{FinAmb}(\mathbb{F}, \{c\})$ as well.

Proof. If $\mathcal{A}$ is a finitely ambiguous weighted automaton over $\mathbb{F}$ and $\{c\}$ realising r, then $h(r)$ is realised by an automaton $\mathcal{B}$ obtained from $\mathcal{A}$ by replacing each transition on c by a sequence of k transitions on c connecting the same pair of states, the intermediate states of this sequence being new and different for each original transition; one of the transitions in the sequence has the same weight as the original transition of $\mathcal{A}$, and the remaining transitions are weighted by 1. It is clear that the characterisation of Theorem 3.2 cannot be spoiled by this transformation, so $\mathcal{B}$ is necessarily finitely ambiguous as well: $h(r) \in \mathsf{FinAmb}(\mathbb{F}, \{c\})$. $\square$

5. Series over Unary Alphabets

Let us start our actual considerations by exploring the relations between the expressive power of finitely sequential, finitely ambiguous, and polynomially ambiguous weighted automata over fields and alphabets containing one single letter – and also by studying strictness of the inclusions between particular levels of the finite ambiguity hierarchy and the finite sequentiality hierarchy in the setting of unary alphabets. Corresponding results for automata over alphabets with at least two letters are gathered in Section 6.

5.1. Finite Sequentiality vs. Finite Ambiguity

We first prove that the classes of finitely sequential and finitely ambiguous automata are equally powerful over unary alphabets – in fact, this result holds not only over fields, but also over an arbitrary commutative semiring.

Given the characterisations of polynomially and finitely ambiguous trim unary weighted automata provided by Theorem 3.2, the number of strongly connected components taking the form of cycles becomes a natural measure of their structural complexity. In order to arrive at the main observation of this section, we need to consider this measure in some detail.

Definition 5.1. Let S be a semiring, $\mathcal{A}$ a trim polynomially ambiguous unary weighted automaton over S and $\Sigma = \{c\}$, and $k \in \mathbb{N}$. We say that $\mathcal{A}$ is a *k-cycle automaton* if it contains at most k directed cycles.

It is easy to see that $\mathcal{A}$ as above is a k-cycle automaton if and only if the algebraic multiplicity of 1 as an eigenvalue of its enumeration matrix is at most k. We mostly apply this measure to *finitely ambiguous* automata in what follows; nevertheless, note that this measure is incomparable with the ambiguity degree in general.

We now note that every trim finitely ambiguous k-cycle automaton $\mathcal{A}$ over a unary alphabet decomposes, for $k \geq 1$, into k finitely ambiguous 1-cycle automata. This result is not supposed to be original, as similar – and often much more nontrivial – decomposition theorems are well known in literature: for instance, every finitely ambiguous weighted automaton can be decomposed, regardless of the alphabet, into a finite union of unambiguous automata [25]. However, as a decomposition result formulated in terms of the number of cycles in unary finitely ambiguous automata does not seem to directly appear anywhere else, we describe the corresponding construction in full detail to make the presentation self-contained.

This construction is in fact intuitively obvious: for each of the cycles, we make use of the criterion (*ii*) of Theorem 3.2, and alter the original automaton $\mathcal{A}$ in order to obtain a 1-cycle automaton, whose successful runs are exactly the successful runs of $\mathcal{A}$ visiting at least one state on the cycle in question. Then we only need to take care of the runs of $\mathcal{A}$ that do not visit any cycle – but these can clearly be realised by a 0-cycle automaton, which may be "adjoined" to any of the k automata without spoiling their 1-cycle property.

Proposition 5.2. *Let S be a semiring, $k \in \mathbb{N} \setminus \{0\}$, and $\mathcal{A}$ a trim finitely ambiguous k-cycle automaton over S and $\Sigma = \{c\}$. Then there are trim 1-cycle automata $\mathcal{A}_1, \ldots, \mathcal{A}_k$ over S and Σ such that $\mathcal{R}_s(\mathcal{A}) = \mathcal{R}_s(\mathcal{A}_1) \uplus \ldots \uplus \mathcal{R}_s(\mathcal{A}_k)$, the values of successful runs of $\mathcal{A}_1, \ldots, \mathcal{A}_k$ being the same as in the original automaton $\mathcal{A}$. This in particular implies that for all $t \in \mathbb{N}$,*

$$\left(\|\mathcal{A}\|, c^t\right) = \sum_{j=1}^{k} \left(\|\mathcal{A}_j\|, c^t\right)$$

and

$$\operatorname{amb}_{\mathcal{A}}(c^t) = \sum_{j=1}^{k} \operatorname{amb}_{\mathcal{A}_j}(c^t).$$

Proof. Without loss of generality, assume that $\mathcal{A}$ contains *precisely* k cycles.[6] Let $\mathcal{A} = (Q, \sigma, \iota, \tau)$ and let the k cycles of $\mathcal{A}$ correspond to state sets $C_1, \ldots, C_k \subseteq Q$, respectively. Thus, denoting by $Q_0 \subseteq Q$ the set of states that do not belong to any cycle, we obtain $Q = Q_0 \uplus C_1 \uplus \ldots \uplus C_k$. For $j = 1, \ldots, k$, denote by $\mathcal{R}_s^{(j)}(\mathcal{A})$ the set of all successful runs of $\mathcal{A}$ visiting at least one state of C_j, *i.e.*,

$$\mathcal{R}_s^{(j)}(\mathcal{A}) = \{\gamma \in \mathcal{R}_s(\mathcal{A}) \mid Q(\gamma) \cap C_j \neq \emptyset\},$$

where $Q(\gamma)$ is the set of states passed by γ, *i.e.*, $Q(\gamma) = \{q_0, \ldots, q_t\}$ for each $\gamma = q_0 c q_1 c q_2 \ldots q_{t-1} c q_t \in \mathcal{R}(\mathcal{A})$ with $q_0, \ldots, q_t \in Q$. For

$$\mathcal{R}_s^{(0)}(\mathcal{A}) = \{\gamma \in \mathcal{R}_s(\mathcal{A}) \mid Q(\gamma) \cap C_j = \emptyset \text{ for } j = 1, \ldots, k\},$$

we clearly obtain $\mathcal{R}_s(\mathcal{A}) = \mathcal{R}_s^{(0)}(\mathcal{A}) \uplus \mathcal{R}_s^{(1)}(\mathcal{A}) \uplus \ldots \uplus \mathcal{R}_s^{(k)}(\mathcal{A})$.

For $j = 1, \ldots, k$, we may also decompose Q as $Q = Q_{\rightarrow} \uplus C_j \uplus Q_{\leftarrow} \uplus Q_{\times}$, where $Q_{\rightarrow}$ consists of all $q \in Q \setminus C_j$ *from which* there exists a run to a state in C_j, $Q_{\leftarrow}$ consists of all $q \in Q \setminus C_j$ *to which* there exists a run from some state in C_j, and $Q_{\times} = Q \setminus (Q_{\rightarrow} \cup C_j \cup Q_{\leftarrow})$. Denote by Q_0' the set of all states $q \in Q_0$ such that $q \in Q(\gamma)$ for some run $\gamma \in \mathcal{R}_s^{(0)}(\mathcal{A})$. Let $Q_j = Q_0' \cup Q_{\rightarrow} \cup C_j \cup Q_{\leftarrow}$ if $j = 1$ and $Q_j = Q_{\rightarrow} \cup C_j \cup Q_{\leftarrow}$ otherwise. Let $\mathcal{A}_j = (Q_j, \iota_j, \sigma_j, \tau_j)$ be a weighted automaton over S and $\Sigma = \{c\}$ such that for all $p, q \in Q_j$,

$$\iota_j(q) = \begin{cases} \iota(q) & \text{if } q \in Q_{\rightarrow} \cup C_j \text{ or } j = 1, \\ 0 & \text{otherwise,} \end{cases}$$

$$\sigma_j(p, c, q) = \begin{cases} \sigma(p, c, q) & \text{if } p \notin Q_{\rightarrow},\ q \notin Q_{\leftarrow}, \text{ or } j = 1 \\ 0 & \text{otherwise,} \end{cases}$$

$$\tau_j(q) = \begin{cases} \tau(q) & \text{if } q \in C_j \cup Q_{\leftarrow} \text{ or } j = 1, \\ 0 & \text{otherwise.} \end{cases}$$

Then $\mathcal{A}_j$ is clearly a trim 1-cycle automaton for $j = 1, \ldots, k$. Moreover, obviously

$$\mathcal{R}_s(\mathcal{A}_1) = \mathcal{R}_s^{(0)}(\mathcal{A}) \uplus \mathcal{R}_s^{(1)}(\mathcal{A})$$

and

$$\mathcal{R}_s(\mathcal{A}_j) = \mathcal{R}_s^{(j)}(\mathcal{A})$$

for $j = 2, \ldots, k$, so that indeed

$$\mathcal{R}_s(\mathcal{A}) = \mathcal{R}_s(\mathcal{A}_1) \uplus \ldots \uplus \mathcal{R}_s(\mathcal{A}_k),$$

the values of successful runs in $\mathcal{A}_1, \ldots, \mathcal{A}_k$ being clearly the same as in $\mathcal{A}$. □

Let us now turn our attention to unary weighted automata over *commutative* semirings, for which we show that finite ambiguity actually coincides with finite sequentiality.

[6] If $\mathcal{A}$ contains ℓ cycles with $1 \leq \ell < k$, then we obtain in this way a decomposition into ℓ automata $\mathcal{A}_1, \ldots, \mathcal{A}_\ell$, and a decomposition into k automata follows by taking $\mathcal{A}_{\ell+1}, \ldots, \mathcal{A}_k$ empty. If $\ell = 0$, then $\mathcal{A}$ itself can be taken for a 1-cycle automaton $\mathcal{A}_1$, while $\mathcal{A}_2, \ldots, \mathcal{A}_k$ can be empty.

Lemma 5.3. *Let S be a commutative semiring, and $\mathcal{A}$ a trim finitely ambiguous 1-cycle automaton over S and unary alphabet $\Sigma = \{c\}$. Then there is a deterministic weighted automaton $\mathcal{B}$ over S and Σ such that $\|\mathcal{B}\| = \|\mathcal{A}\|$.*

Proof. The observation is trivial when $\mathcal{A}$ contains no cycle. We may thus assume that there is *precisely one* cycle in $\mathcal{A} = (Q, \sigma, \iota, \tau)$. Let $\ell \in \mathbb{N} \setminus \{0\}$ be its length, and $\gamma_C = q_1 c q_2 \ldots q_\ell c q_1$, for some $q_1, \ldots, q_\ell \in Q$, a run of $\mathcal{A}$ on c^ℓ that goes around the cycle exactly once. Then there is $t_0 \in \mathbb{N}$ such that for all $t \geq t_0$, each run γ of $\mathcal{A}$ upon c^t visits q_1 and goes around the cycle from q_1 to q_1 at least $\lfloor (t - t_0)/\ell \rfloor$ times.[7] Such γ first follows some run γ_1 until it visits q_1 for the first time, then goes $\lfloor (t - t_0)/\ell \rfloor$ times around γ_C, and finally follows some run γ_2 from q_1 (the run γ_2 may revisit q_1). Setting $M = \sigma(\gamma_C)$, we get $\sigma(\gamma) = (\sigma(\gamma_1)\sigma(\gamma_2))\, M^{\lfloor (t-t_0)/\ell \rfloor}$.

Now, $|\gamma_1| + |\gamma_2| = t - \ell \lfloor (t - t_0)/\ell \rfloor = t - ((t - t_0) - s) = t_0 + s$, where $s \in \{0, \ldots, \ell - 1\}$ is the remainder after dividing $t - t_0$ by ℓ – in other words, we have $t - t_0 \equiv s \pmod{\ell}$. The set of all possible pairs (γ_1, γ_2) is thus finite for all $s \in \{0, \ldots, \ell - 1\}$ and depends only on s. It thus follows that there are $b_0, \ldots, b_{\ell-1} \in S$ such that for $s = 0, \ldots, \ell - 1$ and all $t \geq t_0$ with $t - t_0 \equiv s \pmod{\ell}$,

$$(\|\mathcal{A}\|, c^t) = b_s M^{\lfloor (t-t_0)/\ell \rfloor}.$$

Moreover, for $t = 0, \ldots, t_0 - 1$, denote by a_t the value $(\|\mathcal{A}\|, c^t)$.

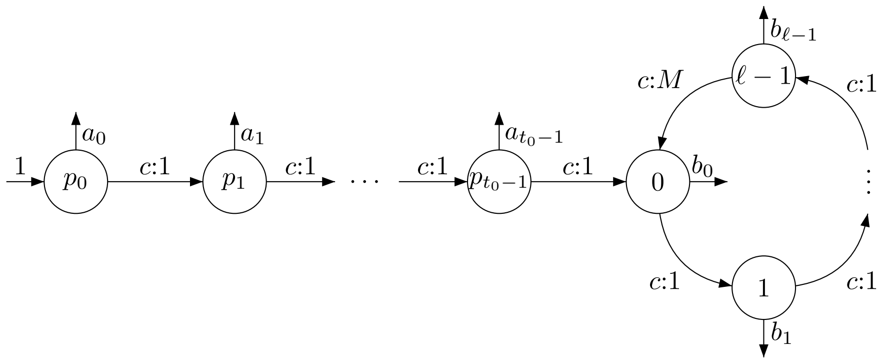


Figure 2: The equivalent deterministic weighted automaton $\mathcal{B}$.

It follows that the automaton $\mathcal{A}$ is equivalent to the deterministic weighted automaton $\mathcal{B}$ over S and $\Sigma = \{c\}$ depicted in Fig. 2. ☐

Remark 5.4. When S is a *field*, the automaton $\mathcal{B}$ from the proof of Lemma 5.3, equivalent to the original automaton $\mathcal{A}$ containing precisely one cycle of length $\ell \in \mathbb{N} \setminus \{0\}$, can be constructed in a slightly more straightforward way. If $\mathcal{P}_\mathcal{A} = (n, \mathbf{i}, A, \mathbf{f})$ with the spectrum of A denoted by $\mathrm{sp}(A)$ and the algebraic multiplicity of each eigenvalue $\lambda \in \mathrm{sp}(A)$ by $\alpha(\lambda)$, one could take $t_0 = \alpha(0)$, and the product of weights of transitions forming the only cycle of $\mathcal{A}$ for M. The automaton $\mathcal{B}$ can then be given in the same way as above, while taking $a_t = (\|\mathcal{A}\|, c^t)$ for $t = 0, \ldots, t_0 - 1$ and $b_s = (\|\mathcal{A}\|, c^{t_0+s})$ for $s = 0, \ldots, \ell - 1$. The matrix B of the linear representation $\mathcal{P}_\mathcal{B}$ then has the same spectrum as A, including multiplicities of the eigenvalues; this in particular implies that $\mathcal{B}$ has precisely n states. As the construction guarantees $(\|\mathcal{B}\|, c^t) = (\|\mathcal{A}\|, c^t)$ for $t = 0, \ldots, n - 1$, the expression (2) has to be the same for both automata, which are thus equivalent as a result.

[7] One can take, e.g., $t_0 = |Q|$.

Theorem 5.5. *Let S be a commutative semiring, $k \in \mathbb{N} \setminus \{0\}$, and $\mathcal{A}$ a trim finitely ambiguous k-cycle automaton over S and unary alphabet $\Sigma = \{c\}$. Then there is a k-sequential weighted automaton $\mathcal{B}$ over S and Σ such that $\|\mathcal{B}\| = \|\mathcal{A}\|$.*

Proof. Decompose $\mathcal{A}$ into trim finitely ambiguous 1-cycle automata $\mathcal{A}_1, \ldots, \mathcal{A}_k$ as in Proposition 5.2, so that $\mathcal{A}_j$ has a deterministic equivalent $\mathcal{B}_j = (Q_j, \sigma_j, \iota_j, \tau_j)$ for $j = 1, \ldots, k$ by Lemma 5.3. Then $\|\mathcal{A}\| = \|\mathcal{B}\|$ for $\mathcal{B}$ the *union* of $\mathcal{B}_1, \ldots, \mathcal{B}_k$, *i.e.*, a k-sequential automaton $\mathcal{B} = (Q, \sigma, \iota, \tau)$ with $Q = (Q_1 \times \{1\}) \cup \ldots \cup (Q_k \times \{k\})$, $\iota(q, j) = \iota_j(q)$, $\sigma((p, j), c, (q, j)) = \sigma_j(p, c, q)$, and $\tau(q, j) = \tau_j(q)$ for all $p, q \in Q$, $j \in [k]$, and $c \in \Sigma$, while $\sigma(p, c, q) = 0$ for all other $(p, c, q) \in Q \times \Sigma \times Q$. □

Corollary 5.6. *Every finitely ambiguous unary weighted automaton $\mathcal{A}$ over a commutative semiring S admits a finitely sequential equivalent* (*and* vice versa)*. Thus* $\mathsf{FinAmb}(S, \{c\}) = \mathsf{FinSeq}(S, \{c\})$.

Proof. If $\mathcal{A}$ is a k-cycle automaton for some $k \in \mathbb{N} \setminus \{0\}$, then it surely admits a k-sequential equivalent by Theorem 5.5. In case $\mathcal{A}$ is a 0-cycle automaton, there clearly is a deterministic weighted automaton equivalent to $\mathcal{A}$. In both cases the automaton equivalent to $\mathcal{A}$ is finitely sequential. Conversely, every finitely sequential automaton is at the same time finitely ambiguous. □

5.2. Finite Ambiguity vs. Polynomial Ambiguity

We now compare the expressive power of finitely and polynomially ambiguous automata over fields and unary alphabets. The inclusion $\mathsf{FinAmb}(\mathbb{F}, \{c\}) \subseteq \mathsf{PolyAmb}(\mathbb{F}, \{c\})$ turns out to be strict if and only if the field $\mathbb{F}$ is of characteristic zero.

C. Barloy et al. [3, 4] have proved that polynomially ambiguous unary weighted automata *over the rationals* are more powerful than their finitely ambiguous counterparts. Let us first observe that their observation directly generalises to all fields of characteristic zero.

Theorem 5.7. *Let $\mathbb{F}$ be a field of characteristic zero. Then there exists a series $r \in \mathbb{F}\langle\langle c^* \rangle\rangle$ realised by a polynomially ambiguous unary weighted automaton over $\mathbb{F}$ and $\Sigma = \{c\}$, but by no finitely ambiguous weighted automaton over $\mathbb{F}$.*

Proof. Let $(r, c^t) = t$ for all $t \in \mathbb{N}$. The series r is then evidently realised by the polynomially ambiguous weighted automaton $\mathcal{A}$ in Fig. 3.

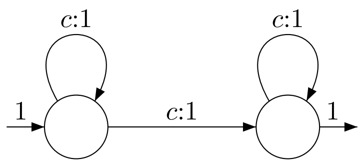


Figure 3: A polynomially ambiguous weighted automaton $\mathcal{A}$ over $\mathbb{F}$ and $\Sigma = \{c\}$ such that $\|\mathcal{A}\| = r$.

Suppose for contradiction that there is a finitely ambiguous automaton realising r. Then it can be decomposed into 1-cycle automata by Proposition 5.2. As $\mathbb{F}$ is of characteristic zero, all polynomials $x^\ell - b$ with $\ell \in \mathbb{N} \setminus \{0\}$ and $b \in \mathbb{F} \setminus \{0\}$ are separable. The nonzero eigenvalues of A are thus of algebraic multiplicity 1 for every 1-cycle automaton $\mathcal{A}$ with $\mathcal{P}_\mathcal{A} = (n, \mathbf{i}, A, \mathbf{f})$. By uniqueness of the expression (2) for (r, c^t), it follows that it cannot contain the term $\binom{t}{1}1^{t-1}$, so that (r, c^t) cannot equal t for all $t \in \mathbb{N}$. □

The situation over fields of positive characteristic turns out to be drastically different, as we now observe.

Theorem 5.8. *Let $\mathbb{F}$ be a field of characteristic $p > 0$ and $\mathcal{A}$ a polynomially ambiguous unary weighted automaton over $\mathbb{F}$ and $\Sigma = \{c\}$. Then there is a finitely ambiguous weighted automaton $\mathcal{B}$ over $\mathbb{F}$ and $\Sigma = \{c\}$ such that $\|\mathcal{B}\| = \|\mathcal{A}\|$.*

Proof. Assume without loss of generality that $\mathcal{A}$ is trim, and let $\mathcal{P}_{\mathcal{A}} = (n, \mathbf{i}, A, \mathbf{f})$ be the corresponding linear representation. By Proposition 3.3,

$$\mathrm{ch}_A(x) = x^{\ell_0} \prod_{k=1}^{s} \left(x^{\ell_k} - b_k\right)$$

for some $\ell_0, s \in \mathbb{N}$, $\ell_1, \ldots, \ell_s \in \mathbb{N} \setminus \{0\}$, and $b_1, \ldots, b_s \in \mathbb{F} \setminus \{0\}$. For convenience, we assume without loss of generality that $\ell_0 > 0$.[8] Let ℓ be the least common multiple of $\ell_0, \ldots, \ell_s$. The spectrum $\mathrm{sp}(A^\ell)$ of the matrix A^ℓ over $\overline{\mathbb{F}}$ is then given by

$$\mathrm{sp}(A^\ell) = \{0\} \cup \left\{ b_k^{\ell/\ell_k} \;\middle|\; k \in [s] \right\} \subseteq \mathbb{F}.$$

For all $\lambda \in \mathrm{sp}(A^\ell)$, let $\alpha(\lambda)$ denote its algebraic multiplicity as an eigenvalue of A^ℓ.

Let $\mathbf{i} = (\iota_1, \ldots, \iota_n)$ and $(\mathbf{e}_1, \ldots, \mathbf{e}_n)$ be the standard (row vector) basis of $\mathbb{F}^n$, *i.e.*,

$$\mathbf{e}_1 = (1, 0, \ldots, 0), \quad \mathbf{e}_2 = (0, 1, \ldots, 0), \quad \ldots, \quad \mathbf{e}_n = (0, 0, \ldots, 1).$$

For $i = 1, \ldots, n$, let $\mathcal{A}_i$ be the automaton $\mathcal{A}$ with initial weight of i changed to 1 and the remaining initial weights changed to 0 – that is, $\mathcal{P}_{\mathcal{A}_i} = (n, \mathbf{e}_i, A, \mathbf{f})$, while clearly

$$\|\mathcal{A}\| = \sum_{i=1}^{n} \iota_i \|\mathcal{A}_i\|.$$

Moreover, for $i = 1, \ldots, n$, let $\mathcal{A}_i^\ell$ be a weighted automaton over $\mathbb{F}$ and $\Sigma = \{c\}$ such that $\mathcal{P}_{\mathcal{A}_i^\ell} = (n, \mathbf{e}_i, A^\ell, \mathbf{f})$. Then for all $t \in \mathbb{N}$,

$$\left(\|\mathcal{A}_i^\ell\|, c^t\right) = \left(\|\mathcal{A}_i\|, c^{\ell t}\right).$$

We first show that the automata $\mathcal{A}_i^\ell$ for $i = 1, \ldots, n$ all admit finitely ambiguous equivalents over $\mathbb{F}$. Let $i \in [n]$ be fixed. The expression (2) for $(\|\mathcal{A}_i^\ell\|, c^t)$ can be written as

$$\left(\|\mathcal{A}_i^\ell\|, c^t\right) = \sum_{\lambda \in \mathrm{sp}(A^\ell)} \sum_{j=0}^{\alpha(\lambda)-1} a_{\lambda,j} \binom{t}{j} \lambda^{t-j} \tag{4}$$

for some constants $a_{\lambda,j} \in \overline{\mathbb{F}}$ for $\lambda \in \mathrm{sp}(A^\ell)$ and $j = 0, \ldots, \alpha(\lambda) - 1$. Moreover, since every $\lambda \in \mathrm{sp}(A^\ell)$ is in $\mathbb{F}$ and $\left(\|\mathcal{A}_i^\ell\|, c^t\right)$ is in $\mathbb{F}$ for all $t \in \mathbb{N}$ as well, the constants $a_{\lambda,j}$ can be determined as a unique solution of a system of linear equations over $\mathbb{F}$ – this means that actually $a_{\lambda,j} \in \mathbb{F}$ for all $\lambda \in \mathrm{sp}(A^\ell)$ and $j = 0, \ldots, \alpha(\lambda) - 1$.

Let us rewrite (4) as

$$\|\mathcal{A}_i^\ell\| = \sum_{\lambda \in \mathrm{sp}(A^\ell)} r_\lambda,$$

where the series $r_\lambda \in \mathbb{F}\langle\langle c^* \rangle\rangle$ is given for each $\lambda \in \mathrm{sp}(A^\ell)$ and all $t \in \mathbb{N}$ by

$$\left(r_\lambda, c^t\right) = \sum_{j=0}^{\alpha(\lambda)-1} a_{\lambda,j} \binom{t}{j} \lambda^{t-j}. \tag{5}$$

[8] This can be assured, for instance, by adding two new states to $\mathcal{A}$, with no ingoing or outgoing transitions, where the first one has initial weight 1 and terminal weight -1, and the second one has both the initial and the terminal weight equal to 1. In case we do not insist on the resulting automaton being trim, the construction is even simpler.

It follows by Proposition 4.1 that in order to prove existence of a finitely ambiguous weighted automaton over $\mathbb{F}$ realising $\|\mathcal{A}_i^\ell\|$, it suffices to prove that every series r_λ with $\lambda \in \mathrm{sp}(A^\ell)$ can be realised by such an automaton.

Let $\lambda \in \mathrm{sp}(A^\ell)$ be fixed. If $\lambda = 0$, the series r_λ is clearly realised by a deterministic weighted automaton, which is at the same time finitely ambiguous. Let us thus assume that $\lambda \neq 0$, let $m \in \mathbb{N} \setminus \{0\}$ satisfy $p^m \geq \alpha(\lambda)$, and let $\mathcal{B}_\lambda = (p^m, \sigma, \iota, \tau)$ be a deterministic 1-cycle weighted automaton over $\mathbb{F}$ and $\Sigma = \{c\}$ given by $\sigma(t, c, t{+}1) = 1$ for $t = 1, \ldots, p^m{-}1$, $\sigma(p^m, c, 1) = \lambda^{p^m}$, $\sigma(t, c, t') = 0$ for all remaining $(t, t') \in [p^m]^2$, $\iota(1) = 1$, $\iota(t) = 0$ for $t = 2, \ldots, p^m$, and $\tau(t) = (r_\lambda, c^{t-1})$ for $t = 1, \ldots, p^m$. If $\mathcal{P}_{\mathcal{B}_\lambda} = (p^m, \mathbf{i}_\lambda, A_\lambda, \mathbf{f}_\lambda)$, then

$$\mathrm{ch}_{A_\lambda}(x) = x^{p^m} - \lambda^{p^m} = (x - \lambda)^{p^m},$$

as $\mathbb{F}$ is of characteristic p. Thus λ is the only eigenvalue of A_λ, and its algebraic multiplicity p^m is at least $\alpha(\lambda)$. The constants in the expression (2) for $\|\mathcal{B}_\lambda\|$ are uniquely determined by $(\|\mathcal{B}_\lambda\|, c^t) = (r_\lambda, c^t)$ for $t = 0, \ldots, p^m - 1$. It follows that the expression (2) for $\|\mathcal{B}_\lambda\|$ is the same as in (5) – in other words, we have $\|\mathcal{B}_\lambda\| = r_\lambda$.

Each of the series r_λ for $\lambda \in \mathrm{sp}(A^\ell)$ is thus realised by a finitely ambiguous weighted automaton over $\mathbb{F}$ and $\Sigma = \{c\}$, and by Proposition 4.1, the same property holds for the series $\|\mathcal{A}_i^\ell\|$. As $i \in [n]$ was arbitrary, there are finitely ambiguous automata for each of the series $\|\mathcal{A}_1^\ell\|, \ldots, \|\mathcal{A}_n^\ell\|$.

Let $h\colon c^* \to c^*$ be a homomorphism given by $h(c) = c^\ell$. Proposition 4.3 then implies that for $i = 1, \ldots, n$, the series $r_{i,\ell} \in \mathbb{F}\langle\langle c^* \rangle\rangle$, given by

$$r_{i,\ell} = h\left(\|\mathcal{A}_i^\ell\|\right),$$

is realised by some finitely ambiguous weighted automaton over $\mathbb{F}$ and $\Sigma = \{c\}$ as well. It is obvious that for all $t \in \mathbb{N}$,

$$\left(r_{i,\ell}, c^t\right) = \begin{cases} (\|\mathcal{A}_i\|, c^t) & \text{if } t \equiv 0 \pmod{\ell}, \\ 0 & \text{otherwise.} \end{cases}$$

Thus, if we denote by $\mathbf{r}_\ell$ the column vector of power series

$$\mathbf{r}_\ell = \left(r_{1,\ell}, \ldots, r_{n,\ell}\right)^T,$$

then clearly

$$\|\mathcal{A}\| = \sum_{j=0}^{\ell-1} \left(\mathbf{i}A^j\, c^j\right) \mathbf{r}_\ell.$$

It thus follows by Proposition 4.2 and Proposition 4.1 that $\|\mathcal{A}\|$ is realised by some finitely ambiguous weighted automaton $\mathcal{B}$ over $\mathbb{F}$ and $\Sigma = \{c\}$ as well. ☐

Corollary 5.9. *Let $\mathbb{F}$ be a field. Then $\mathsf{FinAmb}(\mathbb{F}, \{c\}) \subseteq \mathsf{PolyAmb}(\mathbb{F}, \{c\})$, the inclusion being strict if and only if $\mathbb{F}$ is of characteristic zero.*

Proof. The inclusion $\mathsf{FinAmb}(\mathbb{F}, \{c\}) \subseteq \mathsf{PolyAmb}(\mathbb{F}, \{c\})$ is a consequence of Proposition 2.2, and the rest of the statement follows directly by Theorem 5.7 and Theorem 5.8. ☐

5.3. Infinite Hierarchies

Let us finally consider the infinite hierarchies of series realised, for $k = 0, 1, 2, \ldots$, by the k-ambiguous and k-sequential weighted automata over fields – that is, the *finite ambiguity hierarchy* and the *finite sequentiality hierarchy*. Our aim is to show that these hierarchies are strict if and only if the underlying field is not locally finite, while unary alphabets are sufficient to establish this observation. C. Barloy et al. [3, 4] have noted that the finite ambiguity hierarchy *over the rationals* is strict, describing a counterexample witnessing this fact. We provide a similar counterexample that works over all other than locally finite fields and note that strictness of the finite sequentiality hierarchy is implied by this counterexample as well.

In order to come up with the said counterexample, we use a known fact that other than locally finite fields always contain an element of infinite multiplicative order – recall that this is the case for an element α of a field $\mathbb{F}$ if $\alpha^s = \alpha^t$ for $s, t \in \mathbb{N}$ implies $s = t$.

Proposition 5.10. *Let $\mathbb{F}$ be a field that is not locally finite. Then $\mathbb{F}$ contains an element α of infinite multiplicative order.*

This property actually holds for every other than locally finite commutative semiring [28, Lemma 7.2]; for fields, it also follows by containment of the rational numbers in fields of characteristic zero and by existence of elements transcendental over the Galois field $\mathbb{F}_p$ in other than locally finite fields of characteristic $p > 0$.

We are now prepared to establish the key observation implying strictness of both infinite hierarchies considered in case the underlying field is not locally finite.

Theorem 5.11. *Let $\mathbb{F}$ be a field that is not locally finite and $k \in \mathbb{N}$. Then there exists a series $r \in \mathbb{F}\langle\langle c^* \rangle\rangle$ realised by a $(k+1)$-sequential unary weighted automaton over $\mathbb{F}$ and $\Sigma = \{c\}$, but by no k-ambiguous weighted automaton over $\mathbb{F}$.*

Proof. As $\mathbb{F}$ is not locally finite, Proposition 5.10 guarantees existence of some $\alpha \in \mathbb{F}$ of infinite multiplicative order – that is, $\alpha^s = \alpha^t$ for $s, t \in \mathbb{N}$ implies $s = t$.

Consider a series $r \in \mathbb{F}\langle\langle c^* \rangle\rangle$ given for all $t \in \mathbb{N}$ by

$$\left(r, c^t\right) = \alpha^t + \alpha^{2t} + \ldots + \alpha^{(k+1)t}. \tag{6}$$

Then r is clearly realised by the $(k+1)$-sequential weighted automaton in Fig. 4.

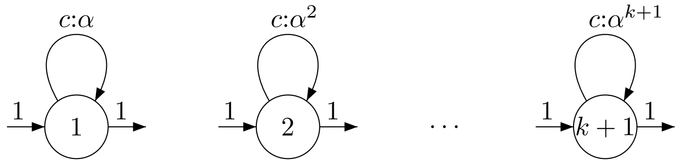


Figure 4: A $(k+1)$-sequential weighted automaton over $\mathbb{F}$ and $\Sigma = \{c\}$ realising the series r.

Suppose for contradiction that the series r is realised by some k-ambiguous weighted automaton $\mathcal{A}$ over $\mathbb{F}$ and $\Sigma = \{c\}$. Without loss of generality, assume $\mathcal{A}$ is trim; moreover, let $\mathcal{P}_{\mathcal{A}} = (n, \mathbf{i}, A, \mathbf{f})$. The spectrum of A then allows us to uniquely express (r, c^t), as a function of $t \in \mathbb{N}$, in the form (2). It thus follows by (6) and Theorem 2.4 that $\alpha, \alpha^2, \ldots, \alpha^{k+1}$ are eigenvalues of A.[9]

As $\mathcal{A}$ is finitely ambiguous, Theorem 3.2 tells us that its strongly connected components are all either directed cycles, or single vertices (without a loop). Nonzero eigenvalues of A are thus precisely the roots of characteristic polynomials of matrices corresponding to the directed cycles, each taking the form $x^\ell - b$ for some $\ell \in \mathbb{N} \setminus \{0\}$ and $b \in \mathbb{F} \setminus \{0\}$.

For each $a \in \overline{\mathbb{F}}$, let $\xi(a)$ be the set of all multiples of a by roots of unity in $\overline{\mathbb{F}}$, *i.e.*,

$$\xi(a) = \left\{\kappa a \mid \kappa \in \overline{\mathbb{F}};\ \exists t \in \mathbb{N} \setminus \{0\} : \kappa^t = 1\right\}.$$

The roots of each polynomial $x^\ell - b$ are then contained in $\xi(a)$ for any of its roots $a \in \overline{\mathbb{F}}$: indeed, if $a, a' \in \overline{\mathbb{F}}$ are roots of $x^\ell - b$, then they are both nonzero and

$$\left(\frac{a'}{a}\right)^\ell = \frac{b}{b} = 1,$$

so that

$$a' = \left(\frac{a'}{a}\right) a \in \xi(a).$$

On the other hand, the sets $\xi(\alpha), \xi(\alpha^2), \ldots, \xi(\alpha^{k+1})$ are pairwise disjoint – if this was not a case, there would exist $x < y \in [k+1]$ such that $\kappa\alpha^x = \nu\alpha^y$ for some roots of unity $\kappa, \nu \in \overline{\mathbb{F}}$; this would imply that $\alpha^{y-x} = \kappa/\nu$ is a root of unity, contradicting the infinite multiplicative order of α. In particular, none of the polynomials $x^\ell - b$ can have two distinct roots among $\alpha, \alpha^2, \ldots, \alpha^{k+1}$. It follows that $\mathcal{A}$ contains $K \geq k+1$ cycles.

[9] Note that $\alpha^{dt} = \binom{t}{0}\left(\alpha^d\right)^{t-0}$ for $d = 1, \ldots, k+1$.

Decompose the automaton $\mathcal{A}$ into 1-cycle automata $\mathcal{A}_1, \ldots, \mathcal{A}_K$ as in Proposition 5.2. For $j = 1, \ldots, K$, let $\mathcal{P}_{\mathcal{A}_j} = (n_j, \mathbf{i}_j, A_j, \mathbf{f}_j)$. Then, by what has been said, $[K] = J_0 \uplus J_1 \uplus \ldots \uplus J_{k+1}$, where J_d consists, for $d = 1, \ldots, k+1$, of precisely all $j \in [K]$ such that the eigenvalues of A_j are in $\xi(\alpha^d) \cup \{0\}$, while they are not all zero; the nonzero eigenvalues of A_j for $j \in J_0$ do not belong to any $\xi(\alpha^d)$ with $d \in [k+1]$. It thus follows by uniqueness of the expression (2) that there exists some $t_0 \in \mathbb{N}$ such that for all $t \geq t_0$,

$$\sum_{j \in J_d} \left(\|\mathcal{A}_j\|, c^t\right) = \alpha^{dt} \qquad \text{for } d = 1, \ldots, k+1.$$

As these values are always nonzero, we find out that the set

$$\bigcup_{j \in J_d} \mathcal{R}_s(\mathcal{A}_j, c^t)$$

is nonempty for $d = 1, \ldots, k+1$, the decomposition of Proposition 5.2 guaranteeing that

$$\mathcal{R}_s(\mathcal{A}, c^t) = \mathcal{R}_s(\mathcal{A}_1, c^t) \uplus \ldots \uplus \mathcal{R}_s(\mathcal{A}_K, c^t).$$

There are thus at least $k+1$ successful runs of $\mathcal{A}$ on c^t, so $\mathcal{A}$ cannot be k-ambiguous: a contradiction. □

The observation just established readily implies strictness both of the finite sequentiality hierarchy and the finite ambiguity hierarchy over other than locally finite fields.

Corollary 5.12. *Let $\mathbb{F}$ be a field. Then for all $k \in \mathbb{N}$, one has $\mathsf{k\text{-}Seq}(\mathbb{F}, \{c\}) \subseteq \mathsf{(k+1)\text{-}Seq}(\mathbb{F}, \{c\})$ and $\mathsf{k\text{-}Amb}(\mathbb{F}, \{c\}) \subseteq \mathsf{(k+1)\text{-}Amb}(\mathbb{F}, \{c\})$, the inclusions for $k \in \mathbb{N} \setminus \{0\}$ being strict if and only if $\mathbb{F}$ is not locally finite, and the inclusion for $k = 0$ being always strict.*

Proof. The inclusions were observed in Proposition 2.2. Let $k \in \mathbb{N}$ be given. When $\mathbb{F}$ is not locally finite, Theorem 5.11 implies existence of a series $r \in \mathbb{F}\langle\langle c^* \rangle\rangle$ that is in $\mathsf{(k+1)\text{-}Seq}(\mathbb{F}, \{c\})$, but not in $\mathsf{k\text{-}Amb}(\mathbb{F}, \{c\})$. By Proposition 2.1, it follows that

$$\begin{aligned} r &\notin \mathsf{k\text{-}Seq}(\mathbb{F}, \{c\}), \qquad & r &\in \mathsf{(k+1)\text{-}Seq}(\mathbb{F}, \{c\}), \\ r &\notin \mathsf{k\text{-}Amb}(\mathbb{F}, \{c\}), \qquad & r &\in \mathsf{(k+1)\text{-}Amb}(\mathbb{F}, \{c\}), \end{aligned}$$

from which strictness of both inclusions follows. In case $\mathbb{F}$ is locally finite, none of the inclusions with $k \in \mathbb{N} \setminus \{0\}$ can be strict by Proposition 2.3 and Proposition 2.2. Moreover, it can be immediately observed that, e.g., the series 1 is both in $\mathsf{1\text{-}Seq}(\mathbb{F}, \{c\}) \setminus \mathsf{0\text{-}Seq}(\mathbb{F}, \{c\})$ and in $\mathsf{1\text{-}Amb}(\mathbb{F}, \{c\}) \setminus \mathsf{0\text{-}Amb}(\mathbb{F}, \{c\})$ for all fields $\mathbb{F}$, locally finite or not. □

6. Series over Alphabets with at Least Two Letters

We now proceed to the case of weighted automata over alphabets containing at least two different letters, for which we consider the same questions as in Section 5 for unary weighted automata.

6.1. Finite Sequentiality vs. Finite Ambiguity

Let us first prove that over alphabets with at least two different letters, the classes of series realised by finitely sequential and finitely ambiguous weighted automata can be separated if and only if the underlying field is not locally finite. This is in sharp contrast with Corollary 5.6, according to which both classes coincide over unary alphabets, regardless of the underlying field.

Of course, Proposition 2.3 implies $\mathsf{FinSeq}(\mathbb{F}, \Sigma) = \mathsf{FinAmb}(\mathbb{F}, \Sigma)$ for all alphabets Σ when $\mathbb{F}$ is locally finite. To give an example of a series witnessing the strict inclusion $\mathsf{FinSeq}(\mathbb{F}, \Sigma) \subsetneq \mathsf{FinAmb}(\mathbb{F}, \Sigma)$ in case $\mathbb{F}$ is not locally finite and Σ contains at least two letters, we use Proposition 5.10 once again.

Theorem 6.1. *Let $\mathbb{F}$ be a field that is not locally finite and $\Sigma = \{a, b\}$. Then there exists a series $r \in \mathbb{F}\langle\langle\Sigma^*\rangle\rangle$ that is realised by an unambiguous weighted automaton over $\mathbb{F}$ and Σ, but by no finitely sequential weighted automaton over $\mathbb{F}$.*

Proof. Let $\alpha \in \mathbb{F}$ be an element of infinite multiplicative order, existence of which follows by Proposition 5.10. Consider the series

$$r = \left(\sum_{t \in \mathbb{N} \setminus \{0\}} \left(a^t b + \alpha^t\, a^t bb\right) \right)^* .$$

This series is clearly realised by an unambiguous automaton $\mathcal{A}$ given as in Fig. 5.

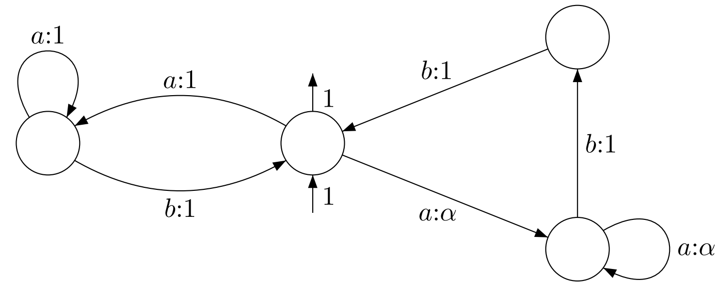


Figure 5: An unambiguous weighted automaton $\mathcal{A}$ over $\mathbb{F}$ and Σ realising r.

We now show that r cannot be realised by a finitely sequential weighted automaton over $\mathbb{F}$. To this end, let us suppose for contradiction that there exists $k \in \mathbb{N} \setminus \{0\}$ and a k-sequential weighted automaton $\mathcal{B} = (n, \sigma, \iota, \tau)$ with $n \in \mathbb{N} \setminus \{0\}$ over $\mathbb{F}$ such that $\|\mathcal{B}\| = r$. Without loss of generality, assume that $k \leq n$.

Given $\|\mathcal{B}\| = r$, it is clear that $\mathcal{B}$ contains at least one cycle composed by transitions on the letter a. Let M be the least common multiple of lengths of all such cycles in $\mathcal{B}$.

Consider an arbitrary vector $\mathbf{s} = (s_1, \ldots, s_n) \in \{1, 2\}^n$ and $d \in \{0, 1, \ldots, M - 1\}$, and let

$$L_{\mathbf{s},d} = \{a^t b^{s_1} a^t b^{s_2} \ldots a^t b^{s_n} \mid t \in \mathbb{N};\ t \geq n;\ t \equiv d \pmod{M}\}.$$

Moreover, let $w_{\mathbf{s},d} \in L_{\mathbf{s},d}$ be chosen in such a way that $\ell := \mathrm{amb}_{\mathcal{B}}(w_{\mathbf{s},d}) \geq \mathrm{amb}_{\mathcal{B}}(w)$ for all $w \in L_{\mathbf{s},d}$. It follows by Proposition 2.1 that ℓ is well-defined and $\ell \leq k \leq n$. Moreover, $(r, w) \neq 0$ for all $w \in L_{\mathbf{s},d}$ implies $\ell \geq 1$. Assume that

$$w_{\mathbf{s},d} = a^{m_{\mathbf{s},d}} b^{s_1} a^{m_{\mathbf{s},d}} b^{s_2} \ldots a^{m_{\mathbf{s},d}} b^{s_n},$$

for some positive integer $m_{\mathbf{s},d}$ no smaller than n such that $m_{\mathbf{s},d} \equiv d \pmod{M}$, and let

$$\gamma_1, \ldots, \gamma_\ell \in \mathcal{R}_s(\mathcal{B}, w_{\mathbf{s},d})$$

be precisely all successful runs of $\mathcal{B}$ upon $w_{\mathbf{s},d}$, finite sequentiality of $\mathcal{B}$ implying that these runs start in pairwise distinct initial states. As $m_{\mathbf{s},d} \geq n$, each run γ_i with $i \in [\ell]$ goes, for $j = 1, \ldots, n$, at least once through some cycle $\gamma_{i,j}$ while reading the j-th factor $a^{m_{\mathbf{s},d}}$ of $w_{\mathbf{s},d}$. Our choice of M implies that $|\gamma_{i,j}|$ divides M for all $i \in [\ell]$ and $j \in [n]$.

For every $\mathbf{t} = (t_1, \ldots, t_n) \in \mathbb{N}^n$ and $i \in [\ell]$, there is a successful run $\gamma_i[\mathbf{t}]$ of $\mathcal{B}$ upon the word

$$w_{\mathbf{s},d}[\mathbf{t}] = a^{m_{\mathbf{s},d} + t_1 M} b^{s_1} a^{m_{\mathbf{s},d} + t_2 M} b^{s_2} \ldots a^{m_{\mathbf{s},d} + t_n M} b^{s_n},$$

which is obtained from γ_i by introducing precisely

$$\frac{t_j M}{|\gamma_{i,j}|}$$

new passes around the cycle $\gamma_{i,j}$ for $j = 1, \dots, n$. As the runs $\gamma_1, \dots, \gamma_\ell$ have pairwise distinct initial states, the runs $\gamma_1[\mathbf{t}], \dots, \gamma_\ell[\mathbf{t}]$ are pairwise distinct. Moreover, there cannot be more than ℓ distinct successful runs of $\mathcal{B}$ on $w_{\mathbf{s},d}[\mathbf{t}]$, as otherwise such runs, necessarily starting in pairwise distinct states of $\mathcal{B}$, would give more than ℓ distinct successful runs of $\mathcal{B}$ on the word $w_{\mathbf{s},d}[\mathbf{t}_{\max}]$, where

$$\mathbf{t}_{\max} = \left(\max_{j \in [n]} t_j, \max_{j \in [n]} t_j, \dots, \max_{j \in [n]} t_j \right);$$

this is impossible by our definition of ℓ, as clearly $w_{\mathbf{s},d}[\mathbf{t}_{\max}] \in L_{\mathbf{s},d}$.

Let $\mathbf{t} = (t_1, t_2, \dots, t_n) \in \mathbb{N}^n$ be fixed and consider the function $f_{\mathbf{s},d,\mathbf{t},1} \colon \mathbb{N} \to \mathbb{F}$ defined for all $t \in \mathbb{N}$ by

$$f_{\mathbf{s},d,\mathbf{t},1}(t) = \left(r, a^{m_{\mathbf{s},d}+tM} b^{s_1} a^{m_{\mathbf{s},d}+t_2 M} b^{s_2} \dots a^{m_{\mathbf{s},d}+t_n M} b^{s_n}\right) = (r, w_{\mathbf{s},d}[\mathbf{t}_{1,t}]),$$

where $\mathbf{t}_{1,t} = (t, t_2, \dots, t_n)$. Our definition of r directly gives

$$f_{\mathbf{s},d,\mathbf{t},1}(t) = C\lambda_{s_1}^t$$

for all $t \in \mathbb{N}$, where $C \in \mathbb{F} \setminus \{0\}$ is some nonzero constant and

$$\lambda_s = \begin{cases} 1 & \text{if } s = 1, \\ \alpha^M & \text{if } s = 2. \end{cases}$$

On the other hand, calculating the values $f_{\mathbf{s},d,\mathbf{t},1}(t)$ for all $t \in \mathbb{N}$ using the runs $\gamma_1[\mathbf{t}_{1,t}], \dots, \gamma_\ell[\mathbf{t}_{1,t}]$ gives

$$f_{\mathbf{s},d,\mathbf{t},1}(t) = \sum_{i=1}^{\ell} g_{i,1}(t), \tag{7}$$

where $g_{i,1} \colon \mathbb{N} \to \mathbb{F}$ is given, for $i = 1, \dots, \ell$ and all $t \in \mathbb{N}$, by

$$g_{i,1}(t) = \overline{\sigma}\left(\gamma_i[\mathbf{t}_{1,t}]\right) = D_i \kappa_{i,1}^t$$

for some $D_i \in \mathbb{F} \setminus \{0\}$ and $\kappa_{i,j} \in \mathbb{F}$ defined for all $i \in [\ell]$ and $j \in [n]$ by

$$\kappa_{i,j} = \sigma(\gamma_{i,j})^{M/|\gamma_{i,j}|}.$$

It then follows by Theorem 2.4 that the functions $g_{i,1}(t)$ on the right-hand side of (7) with $\kappa_{i,1} \neq \lambda_{s_1}$ add up to zero,[10] so that actually

$$f_{\mathbf{s},d,\mathbf{t},1}(t) = \sum_{i \in I(\mathbf{s},d,1)} g_{i,1}(t); \tag{8}$$

here, the set $I(\mathbf{s}, d, j) \subseteq [\ell]$ is defined for all $j \in [n]$ by

$$I(\mathbf{s}, d, j) = \{i \in [\ell] \mid \kappa_{i,j} = \lambda_{s_j}\}.$$

Taking $t = t_1$ in (8), we obtain

$$(r, w_{\mathbf{s},d}[\mathbf{t}]) = \sum_{i \in I(\mathbf{s},d,1)} \overline{\sigma}\left(\gamma_i[\mathbf{t}]\right). \tag{9}$$

Recall that $\mathbf{t} \in \mathbb{N}^n$ is arbitrary, so (9) also holds for all $\mathbf{t} \in \mathbb{N}^n$.

[10] Theorem 2.4 implies that any set of pairwise distinct functions among $\lambda_{s_1}^t = \binom{t}{0}\lambda_{s_1}^{t-0}$ and $\kappa_{i,1}^t = \binom{t}{0}\kappa_{i,1}^{t-0}$ for $i = 1, \dots, \ell$ is linearly independent. The observation then follows by noting that the right-hand side of (7) equals $f_{\mathbf{s},d,\mathbf{t},1}(t) = C\lambda_{s_1}^t$, hence the difference of both should be zero.

Our aim is to strengthen our observation (9) to show that for all $\mathbf{t} \in \mathbb{N}^n$ and $j = 1, \ldots, n$,

$$(r, w_{\mathbf{s},d}[\mathbf{t}]) = \sum_{i \in P(\mathbf{s},d,j)} \overline{\sigma}\left(\gamma_i[\mathbf{t}]\right),$$

where

$$P(\mathbf{s}, d, j) = \bigcap_{h=1}^{j} I(\mathbf{s}, d, h).$$

We prove this by induction on $j = 1, \ldots, n$, the base case (9) with $j = 1$ being already established – indeed, note that $P(\mathbf{s}, d, 1) = I(\mathbf{s}, d, 1)$.

For the induction step, let us suppose that

$$(r, w_{\mathbf{s},d}[\mathbf{t}]) = \sum_{i \in P(\mathbf{s},d,j)} \overline{\sigma}\left(\gamma_i[\mathbf{t}]\right)$$

holds for some $j \in [n-1]$ and all $\mathbf{t} \in \mathbb{N}^n$. Take any fixed vector $\mathbf{t} = (t_1, \ldots, t_n) \in \mathbb{N}^n$, and consider the function $f_{\mathbf{s},d,\mathbf{t},j+1} \colon \mathbb{N} \to \mathbb{F}$ defined for all $t \in \mathbb{N}$ by

$$\begin{aligned} f_{\mathbf{s},d,\mathbf{t},j+1}(t) &= \left(r, a^{m_{\mathbf{s},d}+t_1 M} b^{s_1} \ldots a^{m_{\mathbf{s},d}+t_j M} b^{s_j} a^{m_{\mathbf{s},d}+tM} b^{s_{j+1}} a^{m_{\mathbf{s},d}+t_{j+2} M} b^{s_{j+2}} \ldots a^{m_{\mathbf{s},d}+t_n M} b^{s_n}\right) = \\ &= \left(r, w_{\mathbf{s},d}[\mathbf{t}_{j+1,t}]\right), \end{aligned}$$

where $\mathbf{t}_{j+1,t} = (t_1, \ldots, t_j, t, t_{j+2}, \ldots, t_n)$. Again, our definition of r gives

$$f_{\mathbf{s},d,\mathbf{t},j+1}(t) = C' \lambda_{s_{j+1}}^t$$

for all $t \in \mathbb{N}$ and some constant $C' \in \mathbb{F} \setminus \{0\}$. On the other hand, calculating $f_{\mathbf{s},d,\mathbf{t},j+1}(t)$ for all $t \in \mathbb{N}$ using the induction hypothesis gives

$$f_{\mathbf{s},d,\mathbf{t},j+1}(t) = (r, w_{\mathbf{s},d}[\mathbf{t}_{j+1,t}]) = \sum_{i \in P(\mathbf{s},d,j)} g_{i,j+1}(t), \tag{10}$$

where $g_{i,j+1} \colon \mathbb{N} \to \mathbb{F}$ is given, for all $i \in P(\mathbf{s}, d, j)$ and all $t \in \mathbb{N}$, by

$$g_{i,j+1}(t) = \overline{\sigma}\left(\gamma_i[\mathbf{t}_{j+1,t}]\right) = D_i' \kappa_{i,j+1}^t$$

for some $D_i' \in \mathbb{F} \setminus \{0\}$. Similarly as above, it follows by Theorem 2.4 that the functions $g_{i,j+1}(t)$ on the right-hand side of (10) with $\kappa_{i,j+1} \neq \lambda_{s_{j+1}}$ add up to zero, so that

$$f_{\mathbf{s},d,\mathbf{t},j+1}(t) = \sum_{i \in P(\mathbf{s},d,j) \cap I(\mathbf{s},d,j+1)} g_{i,j+1}(t) = \sum_{i \in P(\mathbf{s},d,j+1)} g_{i,j+1}(t).$$

Taking $t = t_{j+1}$, we finally obtain

$$(r, w_{\mathbf{s},d}[\mathbf{t}]) = \sum_{i \in P(\mathbf{s},d,j+1)} \overline{\sigma}\left(\gamma_i[\mathbf{t}]\right),$$

completing the proof of the induction step.

What we have proved implies in particular that for all $\mathbf{t} \in \mathbb{N}^n$,

$$(r, w_{\mathbf{s},d}[\mathbf{t}]) = \sum_{i \in P(\mathbf{s},d,n)} \overline{\sigma}\left(\gamma_i[\mathbf{t}]\right).$$

As this common value is nonzero, the set $P(\mathbf{s}, d, n)$ is nonempty, which means that there has to be at least one $i \in [\ell]$ such that

$$\kappa_{i,j} = \sigma(\gamma_{i,j})^{M/|\gamma_{i,j}|} = \lambda_{s_j}$$

for $j = 1, \ldots, n$. Let $\gamma_{\mathbf{s},d,\mathbf{t}} := \gamma_i[\mathbf{t}]$; what has been said means that this run on $w_{\mathbf{s},d}[\mathbf{t}]$ goes, for $j = 1, \ldots, n$, through a cycle $\gamma_{\mathbf{s},d,\mathbf{t},j} := \gamma_{i,j}$ satisfying

$$\kappa_{\mathbf{s},d,\mathbf{t},j} := \sigma\left(\gamma_{\mathbf{s},d,\mathbf{t},j}\right)^{M/|\gamma_{\mathbf{s},d,\mathbf{t},j}|} = \lambda_{s_j}$$

while reading the j-th maximal factor from a^+ in $w_{\mathbf{s},d}[\mathbf{t}]$. Note that $\kappa_{\mathbf{s},d,\mathbf{t},j}$ is defined without any ambiguity stemming from different choices of $\gamma_{i,j}$, as a single run of a finitely sequential automaton $\mathcal{B}$ cannot go through two different cycles (understood as subgraphs, as opposed to runs) while reading a factor from a^+.

Let us finally take

$$m = \max\left\{m_{\mathbf{s},d} \mid \mathbf{s} \in \{1,2\}^n;\ d \in \{0, 1, \ldots, M-1\}\right\}$$

and for every $\mathbf{s} \in \{1,2\}^n$, let us consider the word

$$w_{\mathbf{s}} = a^m b^{s_1} a^m b^{s_2} \ldots a^m b^{s_n}.$$

If $d \in \{0, 1, \ldots, M-1\}$ is such that $m \equiv d \pmod{M}$, then for every two distinct vectors $\mathbf{s} = (s_1, \ldots, s_n)$ and $\mathbf{s}' = (s'_1, \ldots, s'_n)$ in $\{1,2\}^n$, one can obviously find $\mathbf{t}, \mathbf{t}' \in \mathbb{N}^n$ such that

$$w_{\mathbf{s}} = w_{\mathbf{s},d}[\mathbf{t}] \qquad \text{and} \qquad w_{\mathbf{s}'} = w_{\mathbf{s}',d}[\mathbf{t}'].$$

Let $j \in [n]$ be the smallest index such that $s_j \neq s'_j$. We have observed above that there is a run $\gamma_{\mathbf{s}} := \gamma_{\mathbf{s},d,\mathbf{t}}$ of $\mathcal{B}$ on $w_{\mathbf{s}}$ that goes through a cycle $\gamma_{\mathbf{s},d,\mathbf{t},j}$ with $\kappa_{\mathbf{s},d,\mathbf{t},j} = \lambda_{s_j}$ while reading the j-th maximal factor from a^+ in $w_{\mathbf{s}}$ for $j = 1, \ldots, n$, as well as a run $\gamma_{\mathbf{s}'} := \gamma_{\mathbf{s}',d,\mathbf{t}'}$ of $\mathcal{B}$ on $w_{\mathbf{s}'}$ that goes through a cycle $\gamma_{\mathbf{s}',d,\mathbf{t}',j}$ with $\kappa_{\mathbf{s}',d,\mathbf{t}',j} = \lambda_{s'_j}$ while reading the j-th maximal factor from a^+ in $w_{\mathbf{s}'}$ for $j = 1, \ldots, n$. As α is of infinite multiplicative order and $s_j \neq s'_j$, surely $\lambda_{s_j} \neq \lambda_{s'_j}$, which means that the runs $\gamma_{\mathbf{s}}$ and $\gamma_{\mathbf{s}'}$ cannot coincide on the common prefix

$$a^m b^{s_1} a^m b^{s_2} \ldots a^m b^{s_{j-1}} a^m$$

of $w_{\mathbf{s}}$ and $w_{\mathbf{s}'}$. It thus follows by finite sequentiality of $\mathcal{B}$ that the runs $\gamma_{\mathbf{s}}$ and $\gamma_{\mathbf{s}'}$ start in different initial states.

As a consequence, we obtain a set

$$\mathcal{R} = \{\gamma_{\mathbf{s}} \mid \mathbf{s} \in \{1,2\}^n\}$$

of precisely 2^n different runs of $\mathcal{B}$ such that every two distinct runs in $\mathcal{R}$ have different initial states. This implies that the number n of states of $\mathcal{B}$ satisfies

$$n \geq 2^n,$$

which is impossible, as $n \in \mathbb{N} \setminus \{0\}$. The series r thus cannot be realised by a finitely sequential weighted automaton over $\mathbb{F}$. ☐

Corollary 6.2. *Let $\mathbb{F}$ be a field and Σ an arbitrary alphabet containing at least two different letters. Then* $\mathsf{FinSeq}(\mathbb{F}, \Sigma) \subseteq \mathsf{FinAmb}(\mathbb{F}, \Sigma)$*, the inclusion being strict if and only if $\mathbb{F}$ is not locally finite.*

Proof. The inclusion was observed in Proposition 2.2. When $\mathbb{F}$ is locally finite, $\mathsf{FinSeq}(\mathbb{F}, \Sigma) = \mathsf{FinAmb}(\mathbb{F}, \Sigma)$ follows by Proposition 2.3 together with Proposition 2.2. If on the other hand $\mathbb{F}$ is not locally finite, one may take any distinct $a, b \in \Sigma$, for which Theorem 6.1 implies existence of a series $r \in \mathsf{UnAmb}(\mathbb{F}, \{a,b\})$ that is not realised by any finitely sequential weighted automaton over $\mathbb{F}$. Thus $r \in \mathsf{FinAmb}(\mathbb{F}, \Sigma) \setminus \mathsf{FinSeq}(\mathbb{F}, \Sigma)$, and the inclusion $\mathsf{FinSeq}(\mathbb{F}, \Sigma) \subseteq \mathsf{FinAmb}(\mathbb{F}, \Sigma)$ is strict. ☐

In addition to implying strictness of the inclusion $\mathsf{FinSeq}(\mathbb{F}, \Sigma) \subseteq \mathsf{FinAmb}(\mathbb{F}, \Sigma)$, Theorem 6.1 says that when $\mathbb{F}$ is not locally finite and Σ contains at least two letters, there actually always exists an *unambiguous* weighted automaton over $\mathbb{F}$ and Σ that does not admit a finitely sequential equivalent. We now complete our understanding of the relations between $\mathsf{UnAmb}(\mathbb{F}, \Sigma)$ and $\mathsf{FinSeq}(\mathbb{F}, \Sigma)$ by noting that a finitely sequential weighted automaton without an unambiguous equivalent exists over all other than locally finite fields $\mathbb{F}$ and all alphabets Σ.

Proposition 6.3. *Let $\mathbb{F}$ be a field that is not locally finite. Then* $\mathsf{UnAmb}(\mathbb{F},\{c\}) \subsetneq \mathsf{FinSeq}(\mathbb{F},\{c\})$*, while* $\mathsf{UnAmb}(\mathbb{F},\Sigma)$ *and* $\mathsf{FinSeq}(\mathbb{F},\Sigma)$ *are incomparable for any alphabet Σ containing at least two different letters.*

Proof. As $\mathbb{F}$ is not locally finite, Theorem 5.11 gives existence of a series $r \in 2\text{-}\mathsf{Seq}(\mathbb{F},\{c\}) \subseteq \mathsf{FinSeq}(\mathbb{F},\{c\})$ that is not in $1\text{-}\mathsf{Amb}(\mathbb{F},\{c\}) = \mathsf{UnAmb}(\mathbb{F},\{c\})$. Thus $r \in \mathsf{FinSeq}(\mathbb{F},\{c\}) \setminus \mathsf{UnAmb}(\mathbb{F},\{c\})$. Moreover, by Proposition 2.2 and Corollary 5.6,

$$\mathsf{UnAmb}(\mathbb{F},\{c\}) \subseteq \mathsf{FinAmb}(\mathbb{F},\{c\}) = \mathsf{FinSeq}(\mathbb{F},\{c\}),$$

so the strict inclusion $\mathsf{UnAmb}(\mathbb{F},\{c\}) \subsetneq \mathsf{FinSeq}(\mathbb{F},\{c\})$ follows.

For an alphabet Σ containing at least two different letters, the same reasoning as above guarantees existence of a series $r_1 \in \mathsf{FinSeq}(\mathbb{F},\Sigma) \setminus \mathsf{UnAmb}(\mathbb{F},\Sigma)$. On the other hand, Theorem 6.1 implies existence of a series $r_2 \in \mathsf{UnAmb}(\mathbb{F},\Sigma) \setminus \mathsf{FinSeq}(\mathbb{F},\Sigma)$. The sets $\mathsf{UnAmb}(\mathbb{F},\Sigma)$ and $\mathsf{FinSeq}(\mathbb{F},\Sigma)$ are incomparable as a result. □

6.2. Finite Ambiguity vs. Polynomial Ambiguity

Let us now continue by examining the relation between $\mathsf{FinAmb}(\mathbb{F},\Sigma)$ and $\mathsf{PolyAmb}(\mathbb{F},\Sigma)$ in case Σ contains at least two different letters. We show that the inclusion $\mathsf{FinAmb}(\mathbb{F},\Sigma) \subseteq \mathsf{PolyAmb}(\mathbb{F},\Sigma)$ is strict whenever the field $\mathbb{F}$ is not locally finite.

Theorem 6.4. *Let $\mathbb{F}$ be a field that is not locally finite and $\Sigma = \{0,1\}$. Then there exists a series $r \in \mathbb{F}\langle\langle\Sigma^*\rangle\rangle$ realised by a polynomially ambiguous weighted automaton over $\mathbb{F}$ and Σ, but by no finitely ambiguous weighted automaton over $\mathbb{F}$.*

Proof. Let $\alpha \in \mathbb{F}$ be an element of infinite multiplicative order, existence of which follows by Proposition 5.10. Consider the series $r \in \mathbb{F}\langle\langle\Sigma^*\rangle\rangle$ defined for all $t \in \mathbb{N}$ and $a_1,\ldots,a_t \in \Sigma$ by

$$(r, a_1 \ldots a_t) = \sum_{k=1}^{t} a_k \alpha^{t-k}.$$

That is, elements of Σ are interpreted as binary digits, and a binary word is evaluated in base α over $\mathbb{F}$ instead of the usual evaluation in base 2 over $\mathbb{Q}$. The series r is clearly realised by the weighted automaton $\mathcal{A}$ over $\mathbb{F}$ and Σ in Fig. 6. It is evident that this automaton is polynomially ambiguous.

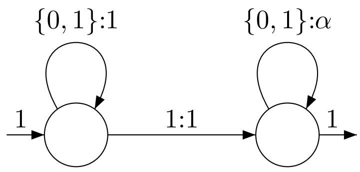


Figure 6: A polynomially ambiguous weighted automaton $\mathcal{A}$ over $\mathbb{F}$ and Σ such that $\|\mathcal{A}\| = r$. An arrow labelled by $\{0,1\}$:β for $\beta \in \mathbb{F}$ should be interpreted as a pair of transitions on 0 and on 1, both weighted by β.

We now prove that the series r cannot be realised by a finitely ambiguous weighted automaton over $\mathbb{F}$. Suppose that the contrary is true, and there is some $k \in \mathbb{N} \setminus \{0\}$ and a k-ambiguous weighted automaton $\mathcal{B} = (n, \sigma, \iota, \tau)$ over $\mathbb{F}$ such that $\|\mathcal{B}\| = r$.

For every $t \in \mathbb{N}$, let us denote by w_t the word

$$w_t = \left(10^t\right)^{k+1},$$

and let

$$L = \{w_t \mid t \in \mathbb{N};\ t \geq n\}.$$

Let $t_0 \in \mathbb{N}$ be such that $t_0 \geq n$ and $\ell := \mathrm{amb}_{\mathcal{B}}(w_{t_0}) \geq \mathrm{amb}_{\mathcal{B}}(w)$ for all $w \in L$. In particular, $\ell \leq k$.

Let us denote the ℓ distinct successful runs of $\mathcal{B}$ on w_{t_0} by $\gamma_1, \ldots, \gamma_\ell$. As $t_0 \geq n$, each of these runs γ_i for $i \in [\ell]$ needs to go, for $j = 1, \ldots, k+1$, around some cycle $\gamma_{i,j}$ while reading the j-th factor 0^{t_0} of w_{t_0}. Similarly as in Theorem 3.2, finite ambiguity of $\mathcal{B}$ implies that $\gamma_{i,j}$ is determined uniquely for each $i \in [\ell]$ and $j \in [k+1]$ as a subgraph, *i.e.*, possibly up to the choice of its initial and terminal state. Let M denote the least common multiple of lengths of these cycles, *i.e.*,

$$M = \operatorname{lcm}\{|\gamma_{i,j}| \mid i \in [\ell];\ j \in [k+1]\}.$$

Then for every $i \in [\ell]$, introducing precisely

$$\frac{tM}{|\gamma_{i,j}|}$$

new passes around the cycle $\gamma_{i,j}$ for $j = 1, \ldots, k+1$ gives rise to a run $\gamma_i[t]$ on the word w_{t_0+tM}, for all $t \in \mathbb{N}$. Moreover, it is not hard to see that in case x is the shortest prefix of w_{t_0} such that runs $\gamma_{i_1}, \gamma_{i_2}$ for some distinct $i_1, i_2 \in [\ell]$ find themselves in two different states p, q after going through x, then there also is a prefix x' of w_{t_0+tM} such that $\gamma_{i_1}[t]$ and $\gamma_{i_2}[t]$ are in these two states p and q after going through x'. The runs $\gamma_1[t], \ldots, \gamma_\ell[t]$ are thus pairwise distinct as well, so that our definition of ℓ implies that these are *precisely all* successful runs of $\mathcal{B}$ upon w_{t_0+tM}.

Now, let us consider a function $f\colon \mathbb{N} \to \mathbb{F}$ given for all $t \in \mathbb{N}$ by

$$f(t) = (r, w_{t_0+tM}).$$

Evaluating this function using our definition of r directly gives

$$f(t) = \sum_{j=1}^{k+1} \alpha^{j(t_0+tM)+j-1} = \sum_{j=1}^{k+1} C_j \left(\alpha^{jM}\right)^t, \tag{11}$$

where the constants $C_j \in \mathbb{F} \setminus \{0\}$ are given, for $j = 1, \ldots, k+1$, by $C_j = \alpha^{j(t_0+1)-1}$. On the other hand, we may also evaluate $f(t)$ using the runs $\gamma_1[t], \ldots, \gamma_\ell[t]$. This gives

$$f(t) = \sum_{i=1}^{\ell} \overline{\sigma}\left(\gamma_i[t]\right) = \sum_{i=1}^{\ell} D_i \lambda_i^t, \tag{12}$$

where for $i = 1, \ldots, \ell$, the constant $D_i \in \mathbb{F} \setminus \{0\}$ is given by $D_i = \overline{\sigma}(\gamma_i)$, while λ_i is given by

$$\lambda_i = \prod_{j=1}^{k+1} \sigma\left(\gamma_{i,j}\right)^{M/|\gamma_{i,j}|}.$$

The elements $\alpha^{1M}, \alpha^{2M}, \ldots, \alpha^{(k+1)M}$ are pairwise distinct as α is of infinite multiplicative order. Moreover, $\ell < k+1$. This means that the right-hand sides of (11) and (12) cannot be the same for all $t \in \mathbb{N}$ by Theorem 2.4. However, the left-hand sides are both equal to $f(t)$ – a contradiction. ☐

Corollary 6.5. *Let $\mathbb{F}$ be a field and Σ an arbitrary alphabet containing at least two different letters. Then* $\mathsf{FinAmb}(\mathbb{F}, \Sigma) \subseteq \mathsf{PolyAmb}(\mathbb{F}, \Sigma)$*, the inclusion being strict if and only if $\mathbb{F}$ is not locally finite.*

Proof. The inclusion was already observed in Proposition 2.2, equality of both sets for locally finite fields follows by Proposition 2.3 and Proposition 2.2, and strictness of the inclusion for other than locally finite fields is implied by Theorem 6.4. ☐

Remark 6.6. It is not hard to see that the polynomially ambiguous automaton $\mathcal{A}$ constructed in the proof of Theorem 6.4 is in fact *linearly ambiguous*, *i.e.*, $\mathrm{amb}_{\mathcal{A}}(w) \leq p(|w|)$ for some *linear* function $p\colon \mathbb{N} \to \mathbb{N}$ and all $w \in \Sigma^*$. The series $r = \|\mathcal{A}\|$ thus actually also separates the classes of series realised by finitely and *linearly* ambiguous weighted automata over $\mathbb{F}$ and Σ, which is a slightly stronger result than the one summarised as Corollary 6.5.

6.3. Infinite Hierarchies

We have already observed strictness of both the finite ambiguity hierarchy and the finite sequentiality hierarchy over unary alphabets whenever the underlying field is not locally finite. This directly implies the same result for arbitrary alphabets.

Theorem 6.7. *Let* $\mathbb{F}$ *be a field and* Σ *an arbitrary alphabet. Then* $\mathsf{k\text{-}Seq}(\mathbb{F},\Sigma) \subseteq \mathsf{(k+1)\text{-}Seq}(\mathbb{F},\Sigma)$ *and* $\mathsf{k\text{-}Amb}(\mathbb{F},\Sigma) \subseteq \mathsf{(k+1)\text{-}Amb}(\mathbb{F},\Sigma)$ *for all* $k \in \mathbb{N}$*, the inclusions for* $k \in \mathbb{N}\setminus\{0\}$ *being strict if and only if* $\mathbb{F}$ *is not locally finite, and the inclusion for* $k = 0$ *being always strict.*

Proof. Strictness of the inclusions when $\mathbb{F}$ is not locally finite or $k = 0$ follows by Corollary 5.12. On the other hand, Proposition 2.3 and Proposition 2.2 imply that the inclusions turn into equalities for $k \in \mathbb{N}\setminus\{0\}$ and $\mathbb{F}$ locally finite. □

7. Conclusions

We have studied the relations between the classes of series realised by the *finitely sequential*, *finitely ambiguous*, and *polynomially ambiguous* weighted automata over fields, as well as the hierarchies composed by the classes of series realised, for $k = 0, 1, 2, \ldots$, by the *k-ambiguous* and *k-sequential* weighted automata over fields – always both over the unary and arbitrary finite alphabets. Using the notation introduced in Section 2, our findings can be summarised as follows.

For every *locally finite* field $\mathbb{F}$, trivially

$$\mathsf{FinSeq}(\mathbb{F},\Sigma) = \mathsf{FinAmb}(\mathbb{F},\Sigma) = \mathsf{PolyAmb}(\mathbb{F},\Sigma)$$

for all alphabets Σ. When $\mathbb{K}$ is a field *of characteristic* $p > 0$ that is *not locally finite*, we still have

$$\mathsf{FinSeq}(\mathbb{K},\{c\}) = \mathsf{FinAmb}(\mathbb{K},\{c\}) = \mathsf{PolyAmb}(\mathbb{K},\{c\})$$

for unary alphabets, while for all alphabets Σ containing at least two different letters,

$$\mathsf{FinSeq}(\mathbb{K},\Sigma) \subsetneq \mathsf{FinAmb}(\mathbb{K},\Sigma) \subsetneq \mathsf{PolyAmb}(\mathbb{K},\Sigma).$$

If finally $\mathbb{L}$ is a field *of characteristic zero*, then

$$\mathsf{FinSeq}(\mathbb{L},\{c\}) = \mathsf{FinAmb}(\mathbb{L},\{c\}) \subsetneq \mathsf{PolyAmb}(\mathbb{L},\{c\})$$

for unary alphabets and

$$\mathsf{FinSeq}(\mathbb{L},\Sigma) \subsetneq \mathsf{FinAmb}(\mathbb{L},\Sigma) \subsetneq \mathsf{PolyAmb}(\mathbb{L},\Sigma)$$

for all alphabets Σ with at least two letters.

Moreover, for the finite sequentiality and finite ambiguity hierarchies over a *locally finite* field $\mathbb{F}$, we trivially observe that

$$\mathsf{0\text{-}Seq}(\mathbb{F},\Sigma) \subsetneq \mathsf{1\text{-}Seq}(\mathbb{F},\Sigma) = \mathsf{2\text{-}Seq}(\mathbb{F},\Sigma) = \mathsf{3\text{-}Seq}(\mathbb{F},\Sigma) = \ldots$$

and

$$\mathsf{0\text{-}Amb}(\mathbb{F},\Sigma) \subsetneq \mathsf{1\text{-}Amb}(\mathbb{F},\Sigma) = \mathsf{2\text{-}Amb}(\mathbb{F},\Sigma) = \mathsf{3\text{-}Amb}(\mathbb{F},\Sigma) = \ldots$$

for all alphabets Σ. Over a field $\mathbb{K}$ that is *not locally finite*, we obtain

$$\mathsf{0\text{-}Seq}(\mathbb{K},\Sigma) \subsetneq \mathsf{1\text{-}Seq}(\mathbb{K},\Sigma) \subsetneq \mathsf{2\text{-}Seq}(\mathbb{K},\Sigma) \subsetneq \mathsf{3\text{-}Seq}(\mathbb{K},\Sigma) \subsetneq \ldots$$

and

$$\mathsf{0\text{-}Amb}(\mathbb{K},\Sigma) \subsetneq \mathsf{1\text{-}Amb}(\mathbb{K},\Sigma) \subsetneq \mathsf{2\text{-}Amb}(\mathbb{K},\Sigma) \subsetneq \mathsf{3\text{-}Amb}(\mathbb{K},\Sigma) \subsetneq \ldots,$$

again for all alphabets Σ.

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