1 Basics

- Mathematics is crucial to computer science, both as a language to express key concepts, and as a tool for solving problems. For instance, machine learning, neural networks, bioinformatics, data science heavily employ vectors and matrices, to the extent that you can't even start doing anything without those. Unlike popular frameworks and specific technologies that easily come out of fashion, what mathematics you learn here will stay the same for the rest of your life.

  `https://docs.google.com/spreadsheets/d/1omEncquM95UKWhO7wrBPd2LEPI3gbINz9SSRBVxTz10/edit?usp=sharing`

- Factorize:

  (a) $t^2 - 4, \quad t^2 - 3, \quad t^2 + 4, \quad x^3 - 4x, \quad a^4 - b^4$

  (b) $u^3 - 1, \quad u^5 - 1, \quad y^5 + 8y^2, \quad a^6 - b^6$

  (c) $d^2 + 5d + 6, \quad 6r^2 - r - 2$

- Simplify (describe plan/reasoning before carrying out manipulations):

  (a) $\frac{a + \sqrt{c}}{a - \sqrt{c}}$, (b) $\frac{x - y}{xy + x^2} \cdot \frac{x^4 - y^4}{y^2 - 2xy + x^2}$, (c) $\frac{\frac{x+y}{x-y} - \frac{x-y}{x+y}}{1 - \frac{x^2+y^2}{x^2-y^2}}$

  (d) $n! \sim \sqrt{2\pi n}\left(\frac{n}{e}\right)^n$; what is $\binom{2n}{n}$? (implications for a row of the Pascal triangle)

- Calculate (using formulas for sums of arithmetic and geometric sequences):

  (a) $1 + 2 + 3 + \cdots + 2n$, (b) $n + (n + 1) + (n + 2) + \cdots + n^2$,

  (c) $1 + 2 + 3 + \cdots + 2^n$, (d) $1 + 2 + 4 + 8 + \cdots + 2^{3n}$,

  (e) $2^2 + 2^6 + 2^{10} + \cdots + 2^{1002}$

2 Polynomials

- Degree and coefficients:
  - Find all quadratic polynomials $P(x)$ such that $P(0) = 0$, $P(1) = 1$, $P(3) = 15$. [$2x^2 - x$]
  - A polynomial of degree n is fully determined by values in $n + 1$ points; coefficients can be found by solving a system of linear equations.
  - Graph of a polynomial depending on its degree; limit behaviour (for large positive/negative numbers).
  - Find all polynomials P such that $3xP(x) = P(x^2) + 1$.
    [Comparison of degrees yields that P has degree at most 1, but we can plug in $a_n x^n + \ldots$; $P(x) = 1/2$.]
- Roots
  - If α is a root of P, then $P(x) = (x - \alpha)Q(x)$ for a suitable polynomial Q.
  - Algorithm for division of polynomials: analogous to integers.
  - Lots of known facts: Every polynomial of odd degree has a real root. Every polynomial has a complex root. If a complex number c is a root, its complex conjugate is also a root. Every real polynomial can be expressed as a product of real polynomials of degree at most 2, or linear complex polynomials. (No need to know it all, but lots of literature to refer to if needed.)

- Quadratic functions
  - The graph of a quadratic function $ax^2 + bx + c$ is a parabola. Find its vertex and minimal value (for $a > 0$).
  - Solve the quadratic inequality $x^2 + 4x - 3 > 0$. (Do not expect "nice" results all the time.)

Additional topics worth to be aware of:

- Vlastnosti funkcií, `https://dml.cz/handle/10338.dmlcz/403455`
- Goniometrické funkcie, `https://dml.cz/handle/10338.dmlcz/403638`
- Komplexné čísla, `https://dml.cz/handle/10338.dmlcz/403625`
- Z. Kubáček o komplexných číslach, `https://www.youtube.com/watch?v=Uw-F0mRX0qE`
- Algebraické rovnice, `https://dml.cz/handle/10338.dmlcz/403551`

3 Logarithms

- Exponential growth: precisely defined, but misused in daily life. Any exponential function with base above 1 grows faster than any polynomial; prove by mathematical induction:
  (a) $4^n > n$ for $n \geq 1$
  (b) $4^n > n^2$ for $n \geq 2$
  (c) $4^n > n^k$ for a fixed k and all sufficiently large n
  [in the induction step, $1 + 1/n$ decreases towards 1 so for large n is lower than $\sqrt[k]{4}$]
- Logarithms
  - definition of logarithm, requirements on basis and argument
  - prove rules from the definition: $\log xy = \log x + \log y$
  - $\log x^n = n \log x, \quad \log(x/y) = \log x - \log y, \quad \log_a x = \log_b x / \log_b a$
- Compute x from the following relations:
  - $\log x = 3 \log m + 2 \log n - 0.5 \log(r + 17)$
  - $6^{3x+2} - 4^{x+3} = 6^{3x+1} + 4^{x+2}$
  - $\log_3(10 \cdot 3^x - 3) - 1 = 2x$
  - $\log_2 x - 2 \log_{1/2} x = 9$
- Z. Kubáček o logaritmoch, `https://www.youtube.com/watch?v=lpFpKXuk2tg`

4 Proofs in number theory

Structure of mathematical proofs: assumptions, conclusion, implications, equivalences. Proof by contradiction (advantage: removes asymmetry between assumptions and the conclusion). Universal quantifier. How to write proofs:

- A proof is essentially a collection of statements and some links between them. Do not be afraid to denote the statements, e.g. (1) or (*), and refer to them explicitly as needed, e.g. "from (1) and (2), we know that . . . (3)".

- Give symbolic names to objects (sets, numbers, functions etc.) and use them in statements. Always define a symbol before you use it. Never use the same letter to denote two different objects.
- Draw pictures. When solving a problem, always ask "what relevant picture can I draw?". In informal proofs, you can use pictures to denote objects easily (and save a ton of words).
- Use ordinary language to describe what you are doing. A sequence of equations is not worth much if you don't explain which equation follows from which other one, or why it follows. Typical proofs contain way more words than you are likely used to. See some proofs in `https://www.emis.de/monographs/Diestel/en/GraphTheoryII.pdf` (don't try to understand them, just ponder the amount of pictures, structure of the sentences, precision of the language, introduction of new symbols etc.).
- Avoid unnecessary things. Saying "we tried blah and failed" is not part of a proof. This does not contradict the previous point about using lots of useful and relevant words. For instance, if the task is to find some object, the solution might be to describe one such object and prove it has all the required properties. It is not necessary to include a story on how we found it, no matter how much mathematics it would include. A textbook would want to include the story, but in your homework, it is likely to be a waste of time.
- Unless the given task is specifically about the manipulations, you can skip common manipulations of expressions (e.g. fraction simplification, equivalent rewriting of linear equations etc.). For instance, if the task is to solve a system of linear equations, you write down the details of your process of solving, but if solving the system is just a small part of a bigger work, you can simply state you solved it by standard means (e.g. WolframAlpha) and write down the solution.
- Typically, you start with the given assumptions, and derive conclusions from them (using well-known facts / theorems along the way), until you arrive at the statement you want to prove. You should refer to general theorems you use by their name or give a specific reference on where to find them. Usually, it also makes sense to refer to recent work done during the lectures ("Použitím postupu z cvičení predmetu Xy dostaneme…").
- In proofs by contradiction, you add the negation of what you want to prove to the assumptions. This results in a set of statements from which you want to derive an obvious falsehood. Don't forget that you get this obvious falsehood from a series of less obvious falsehoods — i.e. whatever statements you get along the way, they likely don't hold in general, only under your specific set of assumptions (which is contradictory).
- In longer proofs, it pays off to split them into a series of lemmas (partial theorems specific to your problem). You formulate a mathematical statement, call it a lemma, give a full proof of it, and later you just refer to the lemma.

Number theory

- $a \mid b$ iff $(\exists k \in \mathbb{Z})(b = ka)$; let them write down definitions of gcd, lcm, coprime numbers
- what is divisible by 0? only 0, use definition — "What it means that $0 \mid x$?" (illustrate that by this question we got rid of one symbol, one layer of complexity by using definition)
- $(a, b) \cdot [a, b] = ab$ (using $d = (a, b)$, by definition $d \mid a$, remove one layer, so $a = Ad$, similarly $b = Bd$, what assumption was not used? d is greatest, so $(A, B) = 1$)
- $\sqrt{2}$ is irrational (proof by contradiction)
- $\sqrt{2} + \sqrt{3} + \sqrt{5}$ is irrational

- existence of prime factorization (proof by induction), it is *unique* — requires additional proof which we skip now
- infinitely many primes (proof by contradiction)
- if a/b is irreducible, then $(a-b)/ab$ is irreducible; does the backward implication hold?
- if $7 \mid 2x+1$, then $7 \mid 3x-2$
- find all pairs of integers x, y such that $x^2 - 5y = 27$
- Euclid's algorithm: $(a,b) = (a-b,b) = (a \bmod b, b)$
- compute $(30, 87)$, $(1001, 91)$, (x^2+1, x), $(x^3+x+2, x+1)$
- divide $x^3 - 6x^2 + 11x - 6$ by $x-4$; compute (x^3-1, x^2-1)

Additional topics from number theory you should be aware of: Euclid's algorithm, congruences. Literature:

- Co víme o přirozených číslech, `https://dml.cz/handle/10338.dmlcz/403433`
- O dělitelnosti čísel celých, `https://dml.cz/handle/10338.dmlcz/403560`
- Kongruence, `https://dml.cz/handle/10338.dmlcz/403650`

5 Mathematical induction

- Prove that $n! < n^n$ for all $n > 1$.
  - Very detailed presentation of a simple proof by induction.
  - What inequalities did we employ in the induction steps?
  - Can the conclusion be derived directly? Non-inductive way often reveals better insight into the problem.
  - Induction in number theory: sometimes natural, other times does not reveal the substance of the problem despite leading to a successful proof — compare the next two problems:
  - Číslo $5^{n+1} + 6^{2n-1}$ je násobkom čísla 31 pre každé prirodzené číslo n.
  - Číslo $a^{5^n} - a$ je násobkom čísla 5 pre každé prirodzené číslo n.
- *Chybný dôkaz:* všetky čísla sú rovnaké (v druhom kroku uvažujeme $(n-1)$-tice rovnakých čísel x_1 až x_{n-1}, x_2 až x_n).
- Prečo sú oba kroky dôležité: nájdite tvrdenie, pre ktoré funguje prvý, ale nefunguje druhý krok indukcie; a naopak (napr. $3^n > 5 \cdot 3^n$; $9 \mid 4^n + 6n + 1$).
- Začiatok indukcie sa dá posunúť podľa potreby: Máme 4- a 7-centové mince (hocikoľko). Dokážte, že nimi vieme vyplatiť ľubovoľnú sumu nad 1 euro.
- Pozor na predpoklady z dokazovaného tvrdenia, musia zostať splnené aj pre menšie objekty vytvorené v druhom kroku: Určte maximálny počet uhlopriečok konvexného n-uholníka, ktoré nemajú žiadne spoločné vnútorné body.
- Pri indukcii sa možno odvolávať na všetky tvrdenia z predošlých krokov, nielen tesne predchádzajúce. Indukciu možno robiť podľa viacerých premenných, alebo podľa inej miery, napr. $m+n$, ak sú v tvrdení dve premenné. Príklady:

  - Pri dôkaze nerovnosti medzi aritmetickým a geometrickým priemerom v druhom kroku $n \to 2n$, $n \to n-1$.
  - Lámeme čokoládu $m \times n$ pozdĺž línií medzi kockami, vždy len jeden kus naraz; najmenej koľkokrát treba lámať, aby sme dostali len kúsky veľkosti 1×1?

- For all positive integers n, prove that

  (a) $5 \mid 7^n - 2^n$, (b) $64 \mid 3^{2n+3} + 40n - 27$,
  (c) $\left(1-\frac{1}{2}\right)\left(1-\frac{1}{3}\right)\cdots\left(1-\frac{1}{n}\right) = \frac{1}{n}$,

  Literature:

  - https://dml.cz/handle/10338.dmlcz/404193
  - https://dml.cz/handle/10338.dmlcz/403889

6 Systematic enumeration of possibilities

- Koľkými spôsobmi sa dá číslo 10 rozložiť na súčet dvoch kladných celých sčítancov, na ktorých poradí nezáleží? A čo pre tri sčítance? [Systematicky podľa veľkosti prvého a potom druhého sčítanca.]
- Nakreslený je štvorec s uhlopriečkami. Koľko trojuholníkov je na obrázku? [Rozdeliť podľa počtu vrcholov štvorca v trojuholníkoch.]
- Nakreslený je (ostrouhlý) trojuholník s ťažnicami. Koľko trojuholníkov je na obrázku? [3 typy bodov: pôvodné vrcholy, stredy strán, ťažisko]
- Koľko je jednoduchých grafov na 4 vrcholoch? [Systematicky nakresliť, tromi spôsobmi: podľa počtu hrán, max. stupňa, dĺžky najdlhšej kružnice. Niekedy sa hodí sekundárne kritérium. Porovnať výhodnosť.]
- Nájdite najmenšie n, pre ktoré je zaručené, že medzi n ľudmi sa určite nájde trojica známych alebo trojica neznámych (vzťahy sú symetrické). [Nakreslite si obrázok. Postupne pre $n = 1, 2, 3, 4, 5$ kreslíme protipríklady. Pre $n = 5$ je to dosť nejasné (aj keby to niekto rýchlo uhádol či poznal), preto chceme systematicky prejsť všetky grafy na 5 vrcholoch, osekávame vetvy, keď sa tam jednofarebná trojica nachádza. Pre $n = 6$ postupujeme systematicky podľa človeka, čo pozná najviac iných, čiže stupňa vrchola.]

7 Inequalities

Main topic: understanding "nested quantifiers", and what it means to find an extreme value of an expression.

- Consider the following two properties regarding a function f on an interval I:
  (1) $\forall x \in I \quad \forall \varepsilon > 0 \quad \exists \delta > 0 \quad \forall y \in I \qquad |x-y| < \delta \implies |f(x) - f(y)| < \varepsilon$
  (2) $\forall \varepsilon > 0 \quad \exists \delta > 0 \quad \forall x \in I \quad \forall y \in I \qquad |x-y| < \delta \implies |f(x) - f(y)| < \varepsilon$
  Determine if $f(x) = 1/x$ has either of these properties on $(0, \infty)$ and prove that the second property implies the first for any f and I.
- Find the largest real number r such that $x^2 - x \geq r$ for all real x.
- Find all real numbers r such that $x^2 - rx + 1 \geq 0$ for all real x.
- Find all real r such that $x^2 + y^2 \geq rxy$ for all positive real x, y.

- Find supremum and infimum of $\frac{x}{x^2+1}$ over (a) positive integers, (b) integers, (c) positive real numbers, (d) non-negative real numbers, (e) real numbers. In which cases are supremum and infimum attained, i.e. they are minimum and maximum?
- Find the minimal value of $x + 1/x$ over (a) $(0, \infty)$, (b) $[3, \infty]$.
- Dokážte, že ak pre reálne čísla x, y platí $x^3 + y^3 \leq 2$, tak $x + y \leq 2$.
[sporom, $y > 2 - x$ dosadíme do prvej nerovnosti]

Literature:

- `https://dml.cz/handle/10338.dmlcz/403599`
- `https://dml.cz/handle/10338.dmlcz/403877`
- `https://dml.cz/handle/10338.dmlcz/403997`

8 Money (discounted cash flow and geometric series)

- If someone offered you 100 EUR today, or 105 EUR in a year, which would you choose? Why? [Reasons for interest to exist: cost of borrowing, balance between borrowers and lenders, time preference of consumption (people prefer to spend today vs. tomorrow), opportunity cost, inflation.]
- If someone offered you to pay 1 EUR a year for 10 years, how much is it worth today? [Given a discount rate, easy to answer in a spreadsheet. Derive a general formula for n periods, payment p, discount rate r.]
- Cash flow streams change in time. How much is a 10-year stream worth if it is growing 3% a year at a 10% discount rate? [Derive a general formula for n periods, initial payment p, growth rate g, discount rate r.]
- How much money do you need for financial independence? How much do you need at a given age if you want to retire at 60? [Use an estimate based on money doubling every 7 years nominally, or every 10 years in real terms.]
- How much does life-long smoking truly cost? About a house. What about one Starbucks coffee a day? [Assume investing at 10% p. a. for 40 years instead of spending the money.]
- What would happen if you only started "frivolous" spending when you are 40, and before that you just invested the money?
- What is the present value of a mortgage at a 10% discount rate? Compare with the value of the property.
- What is the present value of all your future earnings? Compare it with your savings to understand that investing, say, 10 000 euro into a stock index is not a big deal.
- What about an infinite stream of 1 EUR a year? Is it infinitely valuable?
- Derive a general formula for the value of a growing infinite stream (growth rate g, discount rate d).
- Numerical instability: what happens if g approaches d or even exceeds it? Terminal growth rates via an example; e.g. let's say Google is growing at 15% p.a. for 15 years, then 4% p.a. ad infinitum.
- Distant future is kind of irrelevant for stable streams (illustrate on KMI).

- Value of bonds vs. interest rates. How can bonds greatly diminish in value even if they are guaranteed to be repaid in full. Example: Austria 100 year bond issued in 2020, coupon 0.9%, issued at a time when ECB rates were negative 0.5%. Then rates went to 4%. What was the price of the bond at issuance and after the interest rate change? [Discuss risk-less yield and risk premium; duration.]