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Faculty of Mathematics, Physics and Informatics

Decomposition of Small Snarks

Bachelor Thesis

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Univerzita Komenského v Bratislave
Fakulta matematiky, fyziky a informatiky

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Dekompozícia malých snarkov

Cieľ: Analyzovať štruktúru cyklicky 5-súvislých snarkov do 36 vrcholov (s dôrazom na ireducibilné), odhaliť dôvody ich nezafarbiteľnosti a prípadne objavené princípy využiť pri konštrukcii snarkov s danými vlastnosťami.

Anotácia:

Vedúci: RNDr. Ján Mazák, PhD.
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Abstract

A snark is a nontrivial bridgeless cubic graph whose edges can not be colored using three colors. These graphs are important in proving or disproving several famous conjectures of graph theory. G. Brinkmann et al. generated all snarks of order up to 36 using a computer. To better understand the structure and uncolorability of snarks, we analyze all irreducible cyclically 5-edge connected snarks, describe their structure and explain why they are uncolorable. In doing so, we will view a 3-edge-coloring of a cubic graph as at a nowhere zero 4-flow.

In our work, we introduce several operations allowing us to construct new infinite families of snarks.

Keywords: snark, irreducible, cyclical connectivity, Tait coloring, flow

Abstrakt

Snark je netriviálny bezmostový kubický graf, ktorého hrany nemožno zafarbiť tromi farbami. Tieto grafy majú veľký význam v dokazovaní alebo vyvracaní niekoľkých známych hypotéz teórie grafov. G. Brinkmann a kol. vygenerovali pomocou počítača všetky snarky až do rádu 36. Aby sme lepšie pochopili štruktúru a nezafarbiteľnosť snarkov, analyzujeme všetky ireducibilné cyklicky 5-súvislé snarky, opíšeme ich štruktúru a vysvetlíme, prečo sú nezafarbiteľné. Budeme sa pritom pozerať na 3-hranové-zafarbenie kubického grafu ako na nikde nulový 4-tok.

V našej práci predstavíme niekoľko operácií, ktoré nám umožnia konštruovať nekonečné triedy snarkov.

Kľúčové slová: snark, ireducibilný, cyklická súvislosť, Taitovo zafarbenie, tok

Contents

Introduction

The Four Color theorem is one of the most important theorems in graph theory. One of the first attempts to prove it was made by P. G. Tait in 1880 showing that it is equivalent to the statement that every bridgeless cubic planar graph is 3-edge-colorable. At that time, it was believed that every bridgeless cubic graph is 3-edge-colorable. However, this conjecture was disproved by J. Petersen introducing the Petersen graph which is not 3-edge colorable. Since then, the bridgeless cubic graphs that are not 3-edge colorable became the subject of research. M. Gardner named such graphs snarks.

While there was no proof of the Four Color Theorem, snarks were studied in order to prove it or find a counterexample. At present, there are others famous conjectures whose possible counterexamples are snarks, for example the Cycle Double Cover Conjecture, the Five-flow Conjecture or the Fulkerson six 1-factor Double Cover conjecture.

Snarks are very rare graphs. After Petersen published the Petersen graph in 1898, the next two Snark was discovered in 1946 by D. Blanuša [1]. Since 1975 there were only four known snarks, until R. Isaacs discovered two infinite families of snarks [8]. In 2013, G. Brinkmann et al. generated all snarks up to order 36 using a computer [2]. Most of these snarks were not thoroughly analyzed and although we know they are not 3-edge colorable, we do not really understand why.

In our work, we analyze the discovered small snarks and explain why they are uncolorable. Knowing the structure of small snarks is important for further research. Principles observed in small snarks can be generalized to construct infinite families of snarks with specific properties and move us closer to a characterization of all snarks.

Although the essence of the definition of a snark consists of being a bridgeless cubic graph that is not 3-edge-colorable, there can be observed snarks that are more or less trivial modifications of other snarks. Therefore we will focus on snarks which can be considered as *nontrivial* in some reasonable way, namely on the irreducible cyclically 5-connected snarks.

Chapter 1

Multipoles and Snarks

1.1 Multipoles

In our work, we will use the generalized notion of a graph which admits dangling edges, i. e. edges with free ends which we can get after deletion of some edge-cut. Such graphs are called multipoles. The term of the multipole was first used by Fiol [6] and subsequently by other authors [14] [12].

A *multipole* is a pair $M = (V(M), E(M))$, where $V(M)$ is a finite set of vertices and $E(M)$ a finite set od edges. Every edge $e \in E(M)$ has two *ends* and every end of e may or may not be incident with a vertex. The edges of a multipole M are of four types.

1. A *link* is an edge whose ends are incident with two distinct vertices.

2. A *loop* is an edge whose ends are incident with one same vertex.

3. A *dangling edge* is an edge which has only one end incident with vertex.

4. An *isolated edge* is an edge whose both ends are incident with no vertex.

A *semiedge* is an end of an edge that is incident with no vertex. The set of all semiedges of a multipole M is denoted by $S(M)$. Note that a dangling edge contains one semiedge and isolated edge two semiedges. If a multipole has k dangling edges, it is called a k-pole (for instance, a 0-pole is a graph).

Usually, it is convenient to divide the set of the semiedges $S(M)$ into pairwise distinct sets S_1, S_2, $\dots$, S_n which are called *connectors*. Each connector is endowed with a linear ordering of semiedges. A semiedge which is contained in none of the connectors is called a *residual semiedge*. A set of all residual semiedges of a multipole M is denoted by $\mathrm{Res}(M)$. A multipole M with n connectors S_1, S_2, $\dots$, S_n such that $|S_i| = c_i$ for $i \in \{1, 2, \dots, n\}$ and the set of residual semiedges R of the size r is denoted by $M(S_1, S_2, \dots, S_n; R)$ and it is also called a $(c_1, c_2, \dots, c_n; r)$-*pole*. If a connector S

contains only one semiedge s, we will write instead of the set $\{s\}$ only the semiedge s alone.

The *order* of a multipole M, denoted by $|M|$, is the number of its vertices. The *degree* of a vertex v of a multipole is the number of edge ends incident with v and is denoted by $\deg(v)$. In our work, we will consider cubic multipoles, i. e. multipoles where each vertex has degree 3.

An *ordered* multipole is a multipole whose set $S(M)$ of semiedges is endowed with a linear order. If a k-tuple $(e_1, e_2, \dots, e_k)$ represents the linear order of the semiedges of an ordered k-pole M, then we will usually write $M = M(e_1, e_2, \dots, e_k)$.

We will consider a multipole $M(S_1, S_2, \dots, S_k; S_{k+1})$ with the ordered set of residual semiedges S_{k+1} as an ordered multipole with an ordering obtained as the union of orderings of S_1, S_2, $\dots$, S_{k+1} in this order. Note that the order of S_{k+1} can be given implicitly, for example when $|S_{k+1}| \leq 1$.

Next, we describe a method of joining two multipoles together. Let e and f be two edges (not necessary distinct) of a given multipole M and let e, f have semiedges e', f' respectively such that $e' \neq f'$. Then we can identify e with f and construct a new multipole M' in a following way. If $e \neq f$, we replace e and f with a new edge g whose ends are the other ends of e and f. So we set $E(M') = (E(M) - \{e, f\}) \cup \{g\}$. If $e = f$, then e is an isolated edge, and we simply put $E(M') = E(M) - \{e\}$. In this case we in fact create an "isolated loop" which does not have any end and therefore is deleted. In other words, the identification of the semiedges of an isolated edge cancels that edge. We say that the multipole M' arises from M by the *junction* of e' and f'.

Let $M = M(e_1, e_2, \dots, e_k)$ and $N = N(f_1, f_2, \dots, f_k)$ be two ordered k-poles. Then the *junction*, or more precisely the k-junction $M * N$ of M and N, is the graph that arises from the disjoint union $M \cup N$ by performing the junctions e_i with f_i for $i \in \{1, 2, \dots, k\}$. Similarly, for connectors S_1, S_2 of size k of a multipole M, we define the *junction* of the connectors S_1, S_2 as an operation consisting ok k individual junctions of ith semiedge from S_1 and ith semiedge from S_2 for $i \in \{1, 2, \dots, k\}$.

1.2 Tait Colorings of Multipoles

Generally, in our work, we will color multipoles, so also the dangling edges can have color. It is important to choose a convenient set of colors. As we will show later, the set of non-zero elements of the Klein group $\mathbb{Z}_2 \times \mathbb{Z}_2$ has very good properties. We will denote this set as $\mathbb{K}$.

Definition 1. Let M be the multipole and let $\varphi : E(M) \to \mathbb{K}$ be a mapping assigning to each edge of M a color from $\mathbb{K}$. Then φ naturally induces assignment of colors to the edge ends of M. The mapping φ is called a *3-edge-coloring* or simply a *coloring* of the multipole M, if for each vertex $v \in V(M)$ the three edge ends incident with v have assigned pairwise distinct colors.

If there exists a coloring for a multipole M, we say that M is *colorable*, otherwise *uncolorable*.

Using the colors from the set $\mathbb{K}$, we can use addition in the group $\mathbb{Z}_2 \times \mathbb{Z}_2$ to analyze properties of the mapping $\varphi : E(M) \to \mathbb{K}$. If we denote $\delta(v)$ the set of edge ends incident with the vertex v, then obviously φ is a coloring if and only if

$$\sum_{e \in \delta(v)} \varphi(e) = 0$$

for each vertex v. This equation is the Kirchhoff's law for flows in graphs. Thus a coloring of a multipole induces a nowhere-zero $(\mathbb{Z}_2 \times \mathbb{Z}_2)$-flow [5] and vice-versa. Considering that each element in $\mathbb{K}$ is its inverse, we do not have to distinguish the orientation of the flow. Such coloring using the color set $\mathbb{K}$ is also called *Tait coloring.*

When we view a coloring φ of a multipole M as a flow, then we can easily observe form the properties of a flow that

$$\sum_{e \in S(M)} \varphi(e) = 0.$$

Let k_1, k_2 and k_3 be the numbers of dangling edges colored by colors $(0, 1)$, $(1, 0)$ and $(1, 1)$ respectively. If the first entry of the sum of colors from $\mathbb{Z}_2 \times \mathbb{Z}_2$ has to be 0, then k_1 and k_2 have to have the same parity. The same holds for k_2 and k_3. This result is known as the Parity Lemma which was first published by Blanuša [1] and then by Descartes [4], originally stated for the numbers of the used colors in an edge-cut of a 3-edge-colorable graph.

Theorem 1 (Parity Lemma)**.** *Let M be a k-pole and k_1, k_2 and k_3 the numbers of dangling edges colored by color $(0, 1)$, $(1, 0)$ and $(1, 1)$, respectively. Then*

$$k_1 \equiv k_2 \equiv k_3 \equiv k \pmod 2.$$

1.3 Snarks

Before we define snarks, we explain several terms we will use in the characterization of snarks. A very important property of cubic graphs is the connectivity, precisely the edge-connectivity. However, as each cubic graph is at most 3-edge-connected, there is a need to better distinguish the connectivity of cubic graphs. We say that a cubic graph G is *cyclically k-edge-connected* if there is no set S containing less than k edges such that each component of $G - S$ contains a cycle. A *cyclical edge-connectivity* of a graph G is the smallest number k such that G is cyclically k-edge-connected. As the cyclical edge-connectivity and vertex-connectivity of a given graph are equal [13], we will omit the word edge and say only cyclically k-connected and cyclical connectivity.

A *girth* of a graph G is the length of the shortest cycle in G.

In our work, we will use a definition of a snark which follows several famous conjectures which are proposed for bridgeless graphs .

Definition 2. A *snark* is a bridgeless cubic graph which is not 3-edge-colorable.

Many others authors add extra criteria to the definition of snark such as girth at least 5 and high cyclically connectivity to exclude "trivial" snarks. On the other side, there are allowed bridges in snarks in some papers [3] [14]. We will discuss the questions of triviality of snarks later.

The first known snark was found in 1898 by Petersen. It is known as the Petersen graph at present. By 1974, there were only four known snarks when Isaacs discovered the first infinite family of snarks which are named the *flower snarks* or *Isaacs snarks* [8]. Isaacs also discovered an operation allowing to construct a new snark from two given snarks [8].

Definition 3. Let G_1 and G_2 be snarks. The *dot-product* of G_1 and G_2 is a graph $G_1 \circ G_2$ which is constructed in the following way:

1. Subdivide any two nonadjacent edges ab and cd from G_1.

2. Remove any two adjacent vertices u, v from G_2. Let 1, 2 and 3, 4, be the neighbours of v and u, respectively.

3. Join the vertices a, b, c, d to 1, 2, 3, 4 in this order.

Isaacs showed that the dot-product of two snarks is again a snark [8]. We should note that $G_1 \circ G_2$ is formally a set of graphs because the vertices u, v and edges ab, cd can be chosen in multiple ways. For example, the two types of Blanuša's snark [1] are dot-products of two Petersen graph.

With the rise of computers, many snarks was discovered using help of computers. In 2013, Brinkmann et al. generated list of all snarks with order at most 36 [2].

1.4 Questions of Triviality

As snarks often serve as counterexamples, in many definitions of snarks occurs in some form a word "nontrivial". This follows that many snarks are only small modifications of other snarks.

If a snark S contains a triangle, we can replace it by a single vertex as shown in Figure 1.1 resulting in the graph S'. It can be easily shown that the graph S is colorable if and only if S' is colorable. We can look at this from the other side as an operation allowing us to construct infinitely many snarks from a given one, but they will be only the trivial modifications of the former one.

Consider a snark S with a quadrilateral. We can replace it by two parallel edges as in the Figure 1.1 resulting in the graph S'. Again, it is easy to see that if S' is colorable, then S is also colorable. Note that this does not work in the reverse way.

Figure 1.1: Removing triangles and quadrilaterals in a snark.

Now we take a look at snarks with small cyclical connectivity. From the Parity Lemma, it is easy to see that a cubic graph with a bridge is uncolorable.

Consider a snark S with an edge-cut of size 2 which decompose it into two 2-poles M, N. If both M and N are colorable, then from the Parity Lemma, both dangling edges in M and also in N have the same color, so we can extend the colorings of M and N to a coloring of the snark S. Therefore at least one of the components M, N is uncolorable. So we can construct a smaller snark from the snark S by joining the two semiedges in the uncolorable component.

When a snark S has a 3-edge-cut, then again, one of the components M, N has to be uncolorable. Otherwise, the dangling edges of M and N would have three different colors from the Parity Lemma. Thus by connecting the dangling edges of the uncolorable component of S, we can construct a smaller snark.

These were the most common properties of nontrivial snarks which also appear directly in many definitions of snarks. To distinguish these properties we will call snarks trivial and nontrivial.

Definition 4. A snark with girth at least 5 and cyclically edge-conectivity at least 4 will be called *nontrivial snark*. The other snarks will be called *trivial snarks*.

With the conditions guaranteeing nontriviality, we can go further. In 1981, Golberg proved following theorem ([7]).

Theorem 2. *Let G be a snark with a cut-set of four edges whose removal leaves components G_1 and G_2. Then either*

1. *one of G_1 and G_2 is not 3-edge-colorable; or*
2. *both G_1 and G_2 can be "extended" to snarks by adding two vertices to one of them and edges to both in such a way that G is their dot-product.*

This theorem says that every cyclically 4-connected snark arose from one or two smaller snarks. These snarks are not interesting for our work because the cause of their uncolorability lies in other snarks.

Similar result showed Cameron, Chetwynd and Watkins for edge-cuts of size 5 [15]. Nedela and Škoviera generalized theorems of this type for edge-cuts of size k [14].

Theorem 3. *Let G be a snark and let $k \geq 1$ be an integer. Then there exists an integer function $\kappa(k)$ such that if $G = M * N$ is a k-junction of two k-poles, then one of the following statements holds.*

(a) One of M and N is not colorable.

(b) Both M and N can be extended to snarks $\bar{M}$ and $\bar{N}$ by applying the junction with k-poles M' and N', each having at most $\kappa(k)$ vertices. Moreover, $|\bar{M}| \leq |G|$ and $|\bar{N}| \leq |G|$.

Theorem 3, as well as Theorem 2 and other similar theorems of this type, describes two causes which we can observe in edge-cuts of snarks. The part (b) says that the snark G arose from two smaller snarks. Unfortunately, the theorem in general gives us only existence of k-poles M' and N'. It does not describe any way how the multipoles M and N can be extended to snarks or how they created the greater snark G. For $k = 5$, we know at least the the upper bound of added vertices $\kappa(5) = 5$ [15]. However, for $k \geq 6$ remains $\kappa(k)$ unknown.

Now we look at the part (a) of Theorem 3. Let us consider a snark $G = M * N$ satisfying the condition (a), so let us say that M is uncolorable, then M can be extended to a snark $\bar{M} \supseteq M$ of order not greater than $|G|$. In this case, we have reduced snark G to the snark $\bar{M}$ which is called *k-reduction* of G. If additionally $|\bar{M}| < |G|$, then we call such k-reduction *proper*.

If the snark G has any proper k-reduction, the essence of its uncolorability can be found in the smaller snark $\bar{M}$. Thus for the purpose of our work we will aim at snarks which have no proper k-reduction.

Definition 5. A snark is called *k-irreducible* if it has no proper m-reduction for each $m < k$. If a snark is is k-irreducible for each k, then it is called *irreducible.*

Other way of looking on the triviality of snarks is to ask how many and which vertices of a snark we can remove to get a colorable graph. From the Parity Lemma, removing one vertex from a snarks leaves uncolorable graph, so we have to remove at least two vertices. A pair of vertices $\{u, v\}$ of a snark S is called *nonremovable*, if the graph $S - \{u, v\}$ is colorable. If $S - \{u, v\}$ is uncolorable, we call the pair of vertices $\{u, v\}$ *removable.* A snark S is called *critical* if every pair of distinct adjacent vertices in S is nonremovable. Furthermore, if every pair of distinct vertices in S is nonremovable, we call the snark S *bicritical*.

Nedela and Škoviera characterized that a snark is irreducible if and only if it is bicritical. [14].

To sum up, if a snark S admits some k-reduction, we can find the essence of the uncolorability of S to a smaller snarks. Similarly, if the cyclically edge-connectivity of S is smaller than five, the snark S arose from one or two smaller snark. When we exclude mentioned cases, there are cyclically irreducible 5-connected snarks left, which will be the main focus of our work.

1.5 Coloring Sets

To explain the uncolorability of snarks we will study in this work, we will decompose a snark into smaller multipoles and look at the connections between them. Because we will work with irreducible snarks, each multipole alone has to be colorable. One such multipole has several possibilities how its dangling edges can be colored. However, if we look at all multipoles, we would not find any coloring which would assign to all corresponding dangling edges same colors.

Definition 6. Let $M(e_1, e_2, \ldots, e_k)$ be an ordered k-pole. The *coloring set* of the multipole M is the set

$$\mathrm{Col}(M) = \{(\varphi(e_1), \varphi(e_2), \ldots, \varphi(e_k)) \mid \varphi \text{ is a Tait coloring of } M\}.$$

The k-tuple $(\varphi(e_1), \varphi(e_2), \ldots, \varphi(e_k))$ is denoted by $\varphi(S(M))$ for a given coloring φ of the ordered k-pole M with the set of semiedges $S(M)$.

Definition 7. Let $S = (e_1, e_2, \ldots, e_k)$ be a connector of a multipole M. The *flow through* S it the value $\varphi_*(S) = \sum_{i=1}^{k} \varphi(e_i)$. The k-tuple $(\varphi(e_1), \varphi(e_2), \ldots, \varphi(e_k))$ is denoted by $\varphi(S)$.

A connector S is called *proper* if $\varphi_*(S) \neq 0$ for each coloring φ of the multipole M. If $\varphi_*(S) = 0$ for each coloring φ of S, such connector is called *improper*. A multipole is called *proper* if all of its connector are proper and similarly an *improper* multipole has all of its connectors improper.

Two k-poles M and N are called *color-disjoint* if $\mathrm{Col}(M) \cap \mathrm{Col}(N) = \emptyset$. Now we can formally say that to construct a snark, it is sufficient to find two color-disjoint multipoles.

Chapter 2

Commonly Used Multipoles

In many snarks, we can observe several similarities. There are some types of multipoles which occurs in many snarks. They are constructed from snarks by removing some vertices os subdividing some edges. Those multipoles are commonly used in constructing infinite families of snarks with a specific properties [11], [12]. As we study cyclically 5-connected snarks, we will use only multipoles with at least five dangling edges. With the mentioned multipole M we look also on the multipole M' consisting of edges or vertices removed from the snark which M has been constructed from and we describe the coloring set of the multipole M'. This will prove useful in finding other multipoles color-disjoint with M as M' is an example of a color-disjoint multipole.

2.1 Negator

Let S be a snark and uwv a path of length two in S. A *negator* is a $(2, 2; 1)$-pole $M(I, O; R)$ constructed from the snark S in a following way. Remove the u–v–path from S and denote the dangling edges formerly incident with u and not w as e_1, e_2, and the dangling edges formerly incident with v and not w as f_1, f_2. Denote the remaining dangling edge as r. Set $I = (e_1, e_2)$, $O = (f_1, f_2)$ and $R = \{r\}$. We denote the 5-pole constructed in this way as $\mathrm{Neg}(S, u, v)$. Note that this notation might be ambiguous, because there could be more than one common neighbor of the vertices u, v. This implies that the girth of the snark S is at most 4, so the snark S is trivial. As we study primary nontrivial snarks, we will use this notation (used also in [12]) and possible ambiguity will play no significant role in our work.

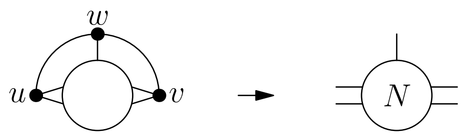

Figure 2.1: The snark S and a symbolic representation of the negator $N = \mathrm{Neg}(S, u, v)$.

For each coloring of a negator $N = \mathrm{Neg}(S, u, v)$, the flow through exactly one of its connector I, O is zero. Otherwise, we could extend such coloring of N to a coloring of the snark S. [12] The flow through the other connector is from the Parity Lemma the same as through the residual semiedge. In other words, the coloring set of N is a subset of

$$C = \{(x, x, a, b, a+b) \mid x, a, b \in \mathbb{K},\ a \neq b\} \cup \{(a, b, x, x, a+b) \mid x, a, b \in \mathbb{K},\ a \neq b\}.$$

A negator whose coloring set is equal to C is called *perfect*, otherwise it is called *imperfect*. For an imperfect negator N, it is possible that one of its connectors is improper, which means the other connector is proper. If such negator N additionally admits all such colorings, it is called *semiperfect*. The following theorem published by Máčajová and Škoviera [12] imply that each negator is either perfect, semiperfect or uncolorable and gives us the characterization of perfect and semiperfect negators.

Theorem 4. *Let $N = \mathrm{Neg}(G, u, v)$ be a colorable negator and w a common neighbor of u and v, then:*

(a) N is perfect if and only if each of the pairs $\{u, w\}$ and $\{v, w\}$ of adjacent vertices is nonremovable,

(b) N is semiperfect if and only if at least one of the pairs $\{u, w\}$ and $\{v, w\}$ is removable.

As we study small snarks, we will mostly use negators obtained from the Petersen graph or Petersen negators, which we denote as N_P. Because of the high symmetry of the Petersen graph, there is up to isomorphism only one way to remove a path of length two from the Petersen graph.

A $(2, 2; 1)$-pole color-disjoint with a negator is obviously a path of length two. It consists of two end vertices u, v and their common neighbor w. We denote it as $P_2(I, O; r)$, where the connector I contains the two semiedges incident with u, O contains the two semiedges incident with v and the residual semiedge r is incident with the vertex w. The coloring set of the P_2 is clearly the set

$$\mathrm{Col}(P_2) = \{(a, b, c, d, e) \in \mathbb{K} \mid a + b \neq 0,\ d + e \neq 0,\ a + b + c + d + e = 0\}.$$

2.2 Proper (2,3)-pole

Let G be a snark, v a vertex in G and e an edge in G. Consider a $(2, 3)$-pole $T(D, E)$ obtained from the snark G in a following way. Remove from G the vertex v and subdivide the edge e. Let D be a set of semiedges arisen from splitting e and E a set of semiedges formerly incident with v.

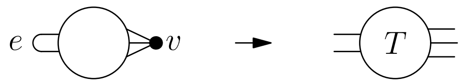

Figure 2.2: The snark G and a symbolic representation of a proper $(2,3)$-pole T.

We claim, that the $(2,3)$-pole T is proper, i. e. for each coloring φ of T, flow through both of the connectors D and E is nonzero. Suppose the contrary. From the Parity Lemma, we get that $\varphi_*(D) = \varphi_*(E) = 0$. Thus we can perform a junction of the two semiedges in D and connect the semiedges in E to a new vertex yielding the snark G. The coloring φ of M can be extended to a coloring of the snark G in a natural way which gives us a contradiction.

Each proper $(2,3)$-pole can be extended to a snark by adding one edge and one vertex in the aforementioned way. In our work, we will consider only proper $(2,3)$-poles that arose from a snark G where the removed vertex v and edge e were not incident, in other words in the constructed $(2,3)$-pole, there will be no vertex incident with two dangling edges which would lead to smaller cyclical connectivity.

The coloring set of each proper $(2,3)$-pole T is a subset of

$$C = \{(a,b,c,d,e) \in \mathbb{K}^5 \mid a+b = c+d+e,\ a+b \neq 0,\ c+d+e \neq 0\}.$$

If T admits all colorings such that flows through both of its connector is non-zero, i. e. its coloring set is equal to C, it is called a *perfect* proper $(2,3)$-pole, otherwise a *imperfect* proper $(2,3)$-pole. An example of an imperfect proper $(2,3)$-pole is given in the section 3.4.

Mostly, we will use proper $(2,3)$-poles from the Petersen graph. Again, due to the high symmetry of the Petersen graph, there is up to isomorphism only one way to remove one vertex v and subdivide an edge not incident with v in the Petersen graph. We denote such proper $(2,3)$-pole constructed from the Petersen graph as T_P. Finding all colorings of T_P using a computer, we observed that T_P is a perfect proper $(2,3)$-pole.

Take a look on the multipole which we removed from the snark G when constructing a proper $(2,3)$-pole. It is a $(2,3)$-pole $M_{ev}(D,E)$ with connectors $D = (d_1, d_2)$, $E = (e_1, e_2, e_3)$, where d_1, d_2 are two ends of an isolated edge and the semiedges e_1, e_2, e_3 are all incident with one common vertex. Its coloring set is

$$\mathrm{Col}(M_{ev}) = \{(x,x,a,b,c) \in \mathbb{K} \mid a+b+c = 0\}.$$

2.3 Odd (2,2,2)-pole

Let G be a snark and v one its vertex with neighbors u_1, u_2 and u_3. Remove from G the vertices v, u_1, u_2, u_3. For $i \in \{1,2,3\}$, denote the semiedges incident with u_i

and not with v as e_i and f_i. Group these semiedges into connectors $S_i = \{e_i, f_i\}$. The resulting $(2,2,2)$-pole $H(S_1, S_2, S_3)$ will be called an *odd* $(2,2,2)$*-pole.*

The name of this multipole is derived from its coloring properties and similar name was used by Goldberg. [7] For each coloring φ of H the number of connectors of H having zero flow is odd. As from the Parity Lemma, it is impossible that flow through exactly two connectors is zero, it is sufficient to show that flow through at least one of the connectors S_1, S_2, S_3 is zero.

Suppose the contrary. If $\varphi_*(S_i) = a_i \neq 0$ for every $i \in \{1,2,3\}$, then we can connect the semiedges e_i and f_i with a new vertex u_i. The color of third semiedge incident with u_i can be set to a_i. From the Parity lemma, $a_1 + a_2 + a_3 = 0$, so all three new semiedges incident with u_1, u_2 and u_3, respectively, have different color, so we can connect them to a new vertex v resulting in the snark G with the coloring φ which is a contradiction.

The simplest example of an odd $(2,2,2)$-pole is a hexagon, a cycle of length six. It arises from the Petersen graph by removing an arbitrary vertex with its neighbors.

Denote the $(2,2,2)$-pole removed from the snark G as $V_4(S_1, S_2, S_3)$. It consist of four vertices v, u_1, u_2 and u_3, where v is the common neighbor of u_1, u_2, u_3. The connector S_i contains the semiedges incident with u_i for $i \in \{1,2,3\}$. The coloring set of V_4 is the set

$$\mathrm{Col}(V_4) = \{(a,b,c,d,e,f) \in \mathbb{K} \mid a \neq b,\ c \neq d,\ e \neq f,\ a+b+c+d+e+f = 0\}.$$

2.4 5-cycle Clusters

Cycles of length five occurs in many snarks and they often are the essence of their uncolorability, as a five-cycle has up to rotation and permutation of colors only one coloring. Therefore, class of multipoles consisting only of five-cycles is of a great importance for our analysis.

A maximal connected subgraph C of a graph G such that each edge of C is contained in some cycle of length five is called a *five-cycle cluster*. As we study small cyclically 5-connected snarks, we will focus on five-cycle clusters that can be constructed from the Petersen graph by removing some vertices or subdividing edges and those five-cycle clusters having at least five dangling edges.

A *pentagon* is the smallest five-cycle cluster, the cycle of length five itself. It has 5 dangling edges. It can be constructed from the Petersen graph by removing a five-cycle.

A *double pentagon* consist of two five-cycles with a common edge, so it has 8 vertices, 9 edges and 6 dangling edges. It arises from the Petersen graph after removing two adjacent vertices and subdividing one edge in such a way that there arises no vertex incident with two semiedges.

A *dyad* consists of two five-cycles with two common adjacent edges. It has 7 vertices, 8 edges and 5 dangling edges. Dyad is the negator constructed from the Petersen graph.

A *triad* is the proper $(2, 3)$-pole obtained from the Petersen graph. It consists of 3 five-cycles, 9 vertices, 11 edges and 5 dangling edges.

Chapter 3

Extensions and Reductions of Snarks

There are several known operations allowing us to construct a new snark from some given one or more snarks. One of these operations is replacing a vertex in a graph S with a triangle giving rise to a graph T which is a snark if and only if S is a snark. Some of these operation are only one-way. Replace a quadrilateral in a snark S with two parallel edges resulting in a graph T. If S is a snark, then so is T, but T can be a snark while S is colorable. There are also less trivial operation like some types of superposition introduced by Kochol [10] where we replace some multipole contained in a given snark with larger one retaining the uncolorability.

The operations of this type can be generalized. We will look at them as at replacing a k-pole M_1 in a graph G_1 with a k-pole M_2 giving rise to a graph G_2. The relation between the colorability of G_1 and G_2 can be derived from the relation between the coloring sets of M_1 and M_2 as shown by Fiol [6].

The k-poles M_1 and M_2 are called *color-equivalent* if $\mathrm{Col}(M_1) = \mathrm{Col}(M_2)$. The k-pole M_1 is said to be *color-contained* in M_2 if $\mathrm{Col}(M_1) \subseteq \mathrm{Col}(M_2)$.

Consider two graphs $G_1 = N * M_1$ and $G_2 = N * M_2$ for k-poles N, M_1 and M_2. Let M_1 be color-contained in M_2. From a coloring of G_1 we can easily obtain a coloring of G_2. So if G_2 is a snark, then G_1 is also a snark. If $|M_1| < |M_2|$ (which imply $|G_1| < |G_2|$), then we say that the multipole M_2 is *color-reducible to* M_1 and that the graph G_1 arose by a *color-reduction of* M_2 *to* M_1. If $|M_1| > |M_2|$, we say that the multipole M_2 is *color-expansible to* M_1 and that the graph G_1 arose from G_2 by a *color-expansion of* M_2 *to* M_1.

The color-expansion is a general method allowing us to construct new snarks from given ones. On the other hand, the color-reduction allow us to move the case of uncolorability to a smaller snark. This general statement alone is useless in constructions of snarks. In order to use it, we need specified pairs of color-contained multipoles. As mentioned beforehand, such examples are a vertex and a triangle which are color-equivalent and two parallel edges are color-contained in a square. Now, we introduce several examples of multipole pairs (M_1, M_2) which we have observed in irreducible cyclically 5-connected snarks.

3.1 The Isaacs 6-poles

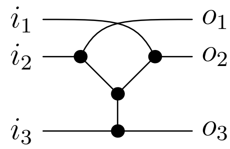

Figure 3.1: $(3,3)$-pole Y used in the construction of the Isaacs flower snarks.

Denote $Y(I, O)$ with $I = (i_1, i_2, i_3)$ and $O = (o_1, o_2, o_3)$ the $(3,3)$-pole shown in Figure 3.1 used in the construction of the Isaacs Flower snarks. Let $Y_k(I_1, O_k)$ denote the $(3,3)$-pole arising from the union of k disjoint copies $Y_i(I_i, O_i)$ of the $(3,3)$-pole Y and performing junctions of the connectors O_i and I_{i+1} for $i \in \{1, 2, \ldots, k-1\}$. Then $\mathrm{Col}(Y_{2m}) = \mathrm{Col}(Y_2)$. This have been observed by Nedela and Škoviera [14].

3.2 The NN 5-pole

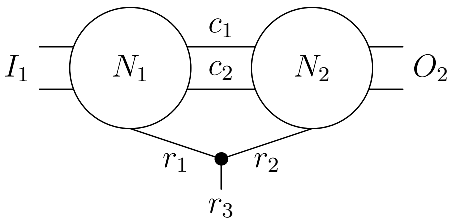

Figure 3.2: A NN $(2, 2; 1)$-pole

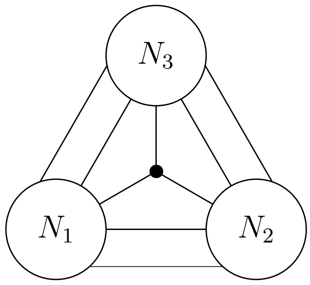

Figure 3.3: A generalized Loupekine snark

Let $N_1(I_1, O_1; r_1)$ and $N_2(I_2, O_2; r_2)$ be two negators. Perform the junction of the connectors O_1 and I_2 and add one vertex v incident with the semiedges r_1, r_2 and one new semiedge denoted by r_3 (see Figure 3.2). Denote the arisen $(2, 2; 1)$-pole $M(I_1, O_2; r_3)$ as $\mathrm{NN}(N_1, N_2)$. We claim that $\mathrm{Col}(\mathrm{NN}(N_1, N_2)) \subseteq \mathrm{Col}(P_2)$. Moreover, if N_1 and N_2 are perfect negators, then $\mathrm{Col}(\mathrm{NN}(N_1, N_2)) = \mathrm{Col}(P_2)$.

Let φ be a coloring of M. Firstly, we observe that flow through the connectors O_1 and I_2 (see Definition 7) has to be zero. Otherwise we would got from the properties of negators that $\varphi(r_1) = \varphi_*(O_1) = \varphi_*(I_2) = \varphi(r_2)$ for adjacent edges r_1, r_2. Then the flows through the others connectors I_1 and O_2 have to be nonzero. This is sufficient to say that $\varphi(S(M)) \in \mathrm{Col}(P_2)$ (see Definition 6).

Suppose that N_1 and N_2 are perfect negators. Let φ be a coloring of dangling edges of M such that $\varphi(S(M)) \in \mathrm{Col}(P_2)$, i. e. $\varphi_*(I_1) = a \neq 0$, $\varphi_*(O_2) = b \neq 0$ and $\varphi(r_3) = a + b \neq 0$. We extend φ to a coloring of the multipole M. We set $\varphi(r_1) = a$, $\varphi(r_2) = b$ getting a zero sum in their common vertex. Furthermore, we set $\varphi(c_1) = \varphi(c_2) = a$ for the edges c_1, c_2 connecting the connectors O_1 and I_2. Both negators N_1, N_2 have admissible colors on their semiedges. As they are perfect, they admit such coloring φ and so does the multipole M which means $\varphi(S(M)) \in \mathrm{Col}(M)$.

We give another view on snarks which are constructed in this way. Let G be a snark which arises form a snark S by a color-expansion of P_2 to $\mathrm{NN}(N_1, N_2)$. The 5-pole S' connected to $\mathrm{NN}(N_1, N_2)$ is a snark S with P_2 removed. It means that S' is a negator $N_3 = \mathrm{Neg}(S, u, v)$ where u, v are the end vertices of P_2. Thus the new snark G has the structure of a Loupekine snark with the Petersen negators replaced with N_1, N_2, N_3 (see Figure 3.3).

3.3 The TT 5-pole

We take two proper $(2,3)$-poles $T_1(D_1, E_1)$, $T_2(D_2, E_2)$ with $E_1 = (e_{11}, e_{12}, e_{13})$, $E_2 = (e_{21}, e_{22}, e_{23})$ and a new vertex v witch we connect to the semiedges e_{13}, e_{23} and denote the remaining semiedge incident with v as r. Perform the junctions of e_{11} and e_{21}, e_{12} and e_{22} (see Figure 3.4). We denote the $(2,2,1)$-pole $M(D_1, D_2, r)$ constructed in this way as $\mathrm{TT}(T_1, T_2)$. We claim that $\mathrm{Col}(\mathrm{TT}(T_1, T_2)) \subseteq \mathrm{Col}(P_2)$. Moreover, if the proper $(2,3)$-poles T_1 and T_2 are perfect, then $\mathrm{Col}(\mathrm{TT}(T_1, T_2)) = \mathrm{Col}(P_2)$.

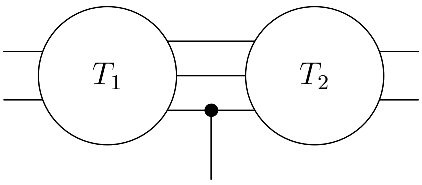

Figure 3.4: A TT $(2,2;1)$-pole

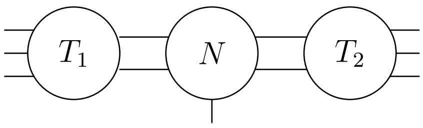

Figure 3.5: A scheme of an uncolorable 7-pole observed in several snarks.

Let φ be a coloring of M. As T_1 and T_2 are proper, we have $\varphi_*(D_1) = a \neq 0$ and $\varphi_*(D_2) = b \neq 0$. Thus again $\varphi(S(M)) \in \mathrm{Col}(P_2)$.

Suppose that T_1 and T_2 are perfect. Let φ be a coloring of the dangling edges of M such that $\varphi(S(M)) \in \mathrm{Col}(P_2)$, i. e. that $a := \varphi_*(D_1) \neq 0 \neq \varphi_*(D_2) =: b$ and $\varphi(r) = a+b \neq 0$. Set $\varphi(e_{13}) = a$, $\varphi(e_{23}) = b$ and $\varphi(e_{11}) = \varphi(e_{12}) = \varphi(e_{21}) = \varphi(e_{22}) = a$.

We have got an admissible coloring of dangling edges of the proper $(2,3)$-poles T_1 and T_2. As they are perfect, they admit such coloring and φ can be extended to a coloring of M.

Take a look on a snark S' which we get after replacing P_2 in a snark S with the 5-pole M. After removing a path of length two in the snark S, we get a negator N. The new snark S' then consists of two proper $(2,3)$-poles T_1, T_2 and one negator N. It contains then the uncolorable 9-pole shown in Figure 3.5.

3.4 The NT 5-pole

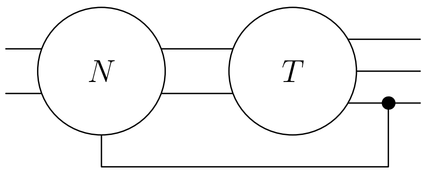

Figure 3.6: A NT $(2,3)$-pole

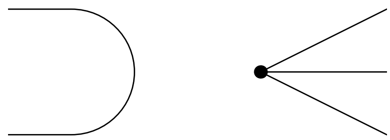

Figure 3.7: A $(2,3)$-pole M_{ev}

Let $N(I,O;r)$ be a negator and $T(D,E)$ a proper $(2,3)$-pole, where $E=(e_1,e_2,f)$. Perform the junction of the connectors O and D and add a new vertex v incident with f, r and a new semiedge e_3 (see Figure 3.6). Denote the resulting $(2,3)$-pole $M(I,J)$, where $J=(e_1,e_2,e_3)$, as $\mathrm{NT}(N,T)$. Take a $(2,3)$-pole M_{ev} described in the section 2.2 (see also Figure 3.7). We show that $\mathrm{Col}(\mathrm{NT}(N,T)) \subseteq \mathrm{Col}(M_{ev})$. Moreover, if N is a perfect negator and T is a perfect proper $(2,3)$-pole, then $\mathrm{Col}(\mathrm{NT}(N,T)) = \mathrm{Col}(M_{ev})$.

Let φ be a coloring of the multipole M. Because T is a proper $(2,3)$-pole, $\varphi_*(D) = \varphi_*(O) \neq 0$ and thus $\varphi_*(I) = 0$. Then the Parity Lemma implies that $\varphi_*(J) = 0$ and therefore $\varphi(S(M)) \in \mathrm{Col}(M_{ev})$.

Suppose that N is a perfect negator and T is a perfect proper $(2,3)$-pole. Let φ be a coloring of the danging edges of M such that $\varphi(J) = (\varphi(e_1), \varphi(e_2), \varphi(e_3)) = (a,b,c)$ for $a+b+c=0$ and $\varphi_*(I)=0$. We set $\varphi(r)=b$, $\varphi(f)=a$ and $\varphi_*(O)=b$. In this way, we have got admissible colorings of the dangling edges of the negator N and the proper $(2,3)$-pole T. As they are both perfect, they admit such coloring φ and so does the multipole M.

After removing M_{ev} from a snark S we get a proper $(2,3)$-pole T'. Thus the color-expansion of M_{ev} to M in a snark S yields a snark consisting of two proper $(2,3)$-poles T, T' and one negator N containing the same uncolorable 9-pole as in the previous section (fig. 3.5).

If we rearranges the semiedges $(i_1,i_2) = I$, $(e_1,e_2,e_3) = J$ of the 5-pole $\mathrm{NT}(N,T)$ in a different way, we obtain an example of a semiperfect negator $M_N(I,(e_1,e_2);e_3)$

with the improper connector I. Also, the $(2,3)$-pole $M_P((i_1, i_2, e_1), (e_2, e_3))$ is proper as $\varphi(i_1)+\varphi(i_2)+\varphi(e_1) = \varphi(e_1) \neq 0$. However, as the semiedges i_1 and i_2 have the same color for each coloring φ of M_P, M_P is an example of an imperfect proper $(2,3)$-pole.

3.5 The TTT 6-pole

We take three proper $(2,3)$-poles $T_i(D_i, E_i)$ and connected them as shown in Figure 3.8. Denote the resulting $(2,2,2)$-pole $M(D_1, D_2, D_3)$ as $\mathrm{TTT}(T_1, T_2, T_3)$. Take the $(2,2,2)$-pole $V_4(S_1, S_2, S_3)$ consisting of one vertex v and three its neighbors u_1, u_2, u_3 (see the section 2.3). Let us show that $\mathrm{Col}(\mathrm{TTT}(T_1, T_2, T_3)) \subseteq \mathrm{Col}(V_4)$. Moreover, if all the proper $(2,3)$-poles T_1, T_2, T_3 are perfect, then $\mathrm{Col}(\mathrm{TTT}(T_1, T_2, T_3)) = \mathrm{Col}(V_4)$.

Let φ be a coloring of M. As D_i is a connector of a proper $(2,3)$-pole, $\varphi_*(D_i) \neq 0$, for $i \in \{1,2,3\}$. This is sufficient to say that $\varphi(S(M)) \in \mathrm{Col}(V_4)$.

Now, suppose that all the proper $(2,3)$-poles are perfect. Let $c_i = \varphi_*(D_i)$ for $i \in \{1,2,3\}$ and $c_1 + c_2 + c_3 = 0$. Let e_1, e_2, e_3 be edges from E_1, E_2, E_3 respectively which are all incident with a common vertex v not contained in any of the proper $(2,3)$-poles. If we assign the same color to the remaining edges from the connectors E_i of the proper $(2,3)$-poles and $\varphi(e_i) = c_i$ for $i \in \{1,2,3\}$, we get admissible coloring of connectors in each proper $(2,3)$-pole, so the whole multipole M is colorable in this way.

Note that after removing the $(2,2,2)$-pole V_4 from a snark S, we get an odd $(2,2,2)$-pole. Thus snark obtained by a color-expansion of V_4 to M consists of an odd $(2,2,2)$-pole connected to the 6-pole $\mathrm{TTT}(T_1, T_2, T_3)$.

3.6 The 3NT 7-pole

Consider three negators $N_i(I_i, O_i; r_i)$ and one proper $(2,3)$-pole $T(E, D)$ and denote $\mathrm{3NT}(N_1, N_2, N_3, T)$ the $(2,2,2;1)$-pole $M(O_1, O_2, O_3; r)$ constructed from the given

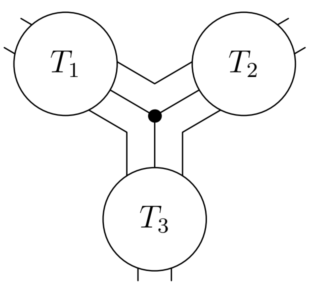

Figure 3.8: A TTT $(2,2,2)$-pole

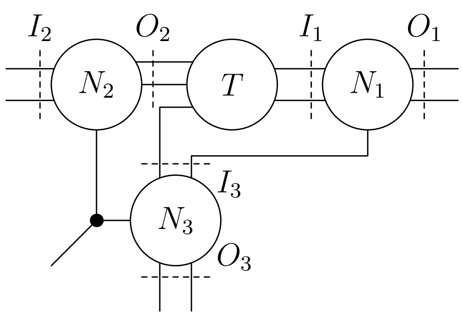

Figure 3.9: A 3NT $(2,2,2;1)$-pole

multipoles as shown in Figure 3.9. Let $M_7((e_1, e_2), I, O; r)$ be a $(2, 2, 2; 1)$-pole obtained by a disjoint union of a path of length two $P_2(I, O; r)$ and an isolated edge with semiedges e_1, e_2. We show that $\mathrm{Col}(M) \subseteq \mathrm{Col}(M_7)$ and if all negators N_1, N_2, N_3 and the proper $(2, 3)$-pole T are perfect, then $\mathrm{Col}(M) = \mathrm{Col}(M_7)$.

Let φ is a coloring of M and let e is the common edge of the connector I_3 and E. As the connector I_1 is connected to the proper $(2, 3)$-pole T, we have $\varphi_*(I_1) = a \neq 0$ and thus $\varphi_*(O_1) = 0$. From the Parity Lemma, we get $\varphi(r_1) = a$.

Suppose that $\varphi_*(I_3) \neq 0$. This means that $\varphi(e) = b \neq a$ and $\varphi(r_3) = a + b$. From the Parity Lemma for the proper $(2, 3)$-pole T, we get that $\varphi_*(I_2) = \varphi(e) + \varphi_*(D) = a+b \neq 0$. As $\varphi_*(I_2) = a+b \neq 0$, from the negator N_1 we get that $\varphi(r_1) = a+b = \varphi(r_2)$, which is a contradiction because the edges r_1 and r_2 have common vertex.

Therefore, $\varphi_*(I_3) = 0$. This implies that $\varphi_*(O_3) \neq 0$. Then, from the Parity Lemma for the proper $(2, 3)$-pole T, we get $\varphi_*(I_2) = a + a = 0$ and thus $\varphi_*(O_2) \neq 0$. Finally, we know that $\varphi(r) \neq 0$ because r is a semiedge. This is sufficient to say that $\varphi(S(M)) \subseteq \mathrm{Col}(M_7)$.

Suppose that all the negators N_1, N_2, N_3 and the proper $(2, 3)$-pole T are perfect. If we take a coloring φ of the dangling edges of M such that $\varphi(S(M)) \in \mathrm{Col}(M_7)$, we can extend it to a coloring of dangling edges of each of the 5-poles N_1, N_2, N_3, T in such a way the flows trough their connectors are the same as in the flows in the proof of the statement $\mathrm{Col}(M) \subseteq \mathrm{Col}(M_7)$ and all multipoles N_1, N_2, N_3, T admit such coloring as they are perfect.

3.7 Pentagon Superposition

A superposition is an process of constructing snarks of grater order from a given snark introduced by Kochol. It can be used to construct snarks with large girth and cyclical connectivity. [10]

A *superedge* is an arbitrary multipole with two connectors and a *supervertex* is an arbitrary vertex with three connectors. Have a cubic graph G and replace each vertex v in G with a supervertex V_v and each edge e in G with a superedge E_e in such a way that if v and e are incident, then some connector of V_v is connected to some connector of E_e with equal size. The graph arisen by this process is called a *superposition* of G.

Let G be a snark. To ensure that a superposition of G is a snark, we have to impose some requirements on used superedges and supervertices. If every edge is proper, the superposition of the snark G is again a snark. [10] However, it is not a necessary condition. We introduce one type of superposition we observed in studied snarks.

Take a look on a proper $(2, 3)$-pole $T(D, E)$ with $D = (d_1, d_2)$. Arranging its

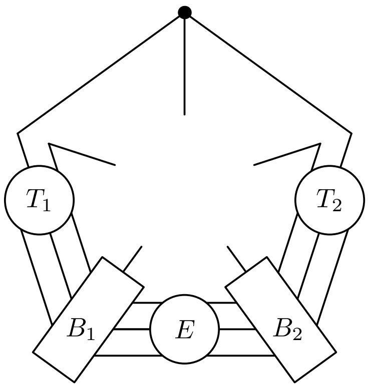

Figure 3.10: A superpentagon used to replace a pentagon in a superposition

semiedges in a different way, we can get a $(3, 1, 1)$-pole $V(E, d_1, d_2)$ which is also proper or in words of superposition, it is a proper supervertex.

As observed by Kochol, a proper superedge can be constructed from a snark S by removing two non-adjacent vertices u, v leaving a $(3, 3)$-pole $E(S_1, S_2)$, where the connectors S_1, S_2 contains the three semiedges formerly incident with u, v respectively. From the Parity Lemma, we get that $\varphi_*(S_1) = \varphi_*(S_2) = a$ for some coloring φ and we also have $a \neq 0$, otherwise we could extend φ to a coloring of the snark S. [10] Consider a special case of this construction. If the distance of the removed vertices is 2, the arisen superedge can be seen as a negator $\mathrm{Neg}(S, u, v)$ with one extra vertex incident with the residual semiedge.

We also use a supervertex $B(S_1, S_2, r)$ used by Kochol with $S_1 = (e_1, f_1, g_1)$ and $S_2 = (e_2, f_2, g_2)$. It consists of one vertex v incident with semiedges e_1, e_2, r and two isolated edges with ends f_1, f_2 and g_1, G_2, respectively. Note that B is not a proper supervertex.

Take a vertex v, two proper supervertices obtained from proper $(2, 3)$-poles T_1, T_2, respectively, two Kochol's supervertices B_1, B_2 and a proper Kochol's superedge E. Connect them cyclically in the order v, T_1, B_1, E, B_2, T_2 (as shown in Figure 3.10) resulting in a 5-pole M. We show that $\mathrm{Col}(M) \subseteq \mathrm{Col}(P_5)$, where P_5 denotes a pentagon. In other words, the superposition of a snark S where we replace a pentagon with the multipole M and leave the remaining edges and vertices unchanged is a snark.

It is sufficient to show, that for each coloring φ of M, each superedge in M has nonzero flow through each of its connectors. Superedges incident with v are casual edges and the superedge E is proper, so they have nonzero flow through both connectors. The $(3, 1, 1)$-pole T_1 is a proper supervertex, so the flow through the superedge incident with the supervertices T_1 and B_1 (consisting of three parallel isolated edges) is nonzero. For the same reason the flow through the semiedge incident with T_2 and B_2 is also non-zero.

Chapter 4

Decomposition of Snarks up to Order 36

In this chapter, we summarize results of our research. Most of the researched snarks can be constructed from the Petersen graph by employing some aforesaid operations.

4.1 Methods of Identification

The key of our analysis is to identify multipoles mentioned in the chapter 2. As we study small snarks, most of those multipoles are taken form the Petersen graph. In other words, they are five-cycle clusters, which are not so hard to identify in a given graph. To do so, we used computer program implemented by Simeunovič [16]. The output of this program contains for each graph G a list of 5-cycle clusters contained in G and for each such cluster C, there is the type of C (dyad, triad, etc.) and list of vertices contained in C.

However, knowing only the vertices contained in a cluster is not sufficient. We need to identify connectors of each cluster, precisely, identify the vertices which are incident with semiedges in the connectors of each cluster. We will call these vertices the *outer vertices* and the remaining vertices of a cluster which are incident with no semiedges the *inner vertices*. We say that the outer vertex v is *adjacent to* a connector S, if v is incident with some semiedge from S.

Take the **dyad** $N_P(I, O; r)$. The outer vertex c adjacent to the residual semiedge r is the only outer vertex adjacent to two inner vertices. The other outer vertices are all vertices in N_P having from c the distance two. We take one arbitrary vertex at distance two from c and denote it a_1. There is only one outer vertex different from c at distance two from a_1, let it be a_2. Denote the remaining two outer vertices, distanced from a_1 one and three respectively, b_1 and b_2. In the end, the vertices a_1, a_2 and also b_1, b_2 are adjacent to the a common connector.

Figure 4.1: The dyad (left) and triad (right) with denoted outer vertices

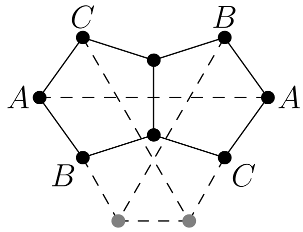

Figure 4.2: The double pentagon with marked connectors adjacent to outer vertices and its construction from the Petersen graph

In the **triad** $T_P(D, E)$ there is only one outer vertex v which is incident only with inner vertices and it is adjacent to the connector E. The vertices adjacent to the connector D are the outer vertices at distance two from v. The remaining two outer vertices adjacent to E can be found as the remaining outer vertices.

In the **double pentagon** $C_{D5}(A, B, C)$ (see Figure 4.2), we firstly identify one vertex from the connector A, let us say v. It is one of the vertices which is incident with no inner vertex. Then staring form the vertex v and the outer vertices along the hamiltonian cycle of the C_{D5} (which is the multipole C_{D5} without the edge incident with the two inner vertices) are adjacent to connectors A, B, C, A, B, C, respectively.

To analyze the **pentagon**, we only need the order of its vertices which can be obtained directly from the graph code.

After identifying five-cycle clusters a given snark S, we can check whether the snark S contains some of the 5-poles described in the chapter. We denote the multipoles which we will identify in a following way: $P_{NN} = \mathrm{NN}(N_P, N_P)$, $P_{NT} = \mathrm{NT}(N_P, T_P)$, $P_{TT} = \mathrm{TT}(T_P, T_P)$ and $P_{TTT} = \mathrm{TTT}(T_P, T_P, T_P)$. All mentioned multipoles consist of Petersen negators N_P and Petersen proper $(2, 3)$-poles T_P.

As we have shown in the previous chapter, the multipoles P_{NN}, P_{NT}, P_{TT} and P_{TTT} are color-equivalent to P_2, M_{ev}, P_2 and V_4, respectively. Suppose we find some of those multipoles in the snark S, denote it M_1 and let $M_2 \in \{P_2, R, V_4\}$ be a multi-pole which is color-equivalent to M_1. Replacing the multipole M_1 with M_2 has to lead us to a smaller snark S'. Therefore because of the color-equivalence, we know that the snark S arose from some smaller snark S' by a color-expansion of M_2 to M_1.

Knowing this, it is sufficient to explain the uncolorability of S. However, it is not hard to determine the snark S' which was the snark S constructed from. As we know the orders of M_1, M_2 and S, we can compute the order of S'. While we study snarks of order at most 36, the order of snark S will not exceed 24. We can construct the graph S', choose the some snark G of the order $|S'|$ and check if it is isomorphic with S'. The choosing of S' is not difficult, as there are only few snarks of order up to 24. We observed, that the snark S' do not have to have the same cyclically edge-connectivity as the snark S and it can be even reducible.

After identifying snarks which arose from smaller snarks by some color-expansion described in the previous chapter, there are some snarks left. They can be divided in several classes. Each class can be characterized by a specific connection of some multipoles. For many classes, small snark contained in them consist of the Petersen five-cycle clusters. Thus for these classes, it is simple to check using a computer if a given snark S belongs to the considered class. We check whether S contains right types of five-cycle clusters and if they are connected to each other in the desired way. For classes consisting of a small number of snarks, we made the analysis by hand.

Although there is only one Petersen negator N_P and only one Petersen proper $(2,3)$-pole with the respect to isomorphism, the order of their semiedges may vary. This can lead to several nonisomporphic variations of the multipoles P_{NN}, P_{NT}, P_{TT} and P_{TTT}. Since the order of semiedges plays no significant role in explaining the uncolorability of snarks, we allow this minor formal impreciseness.

4.2 Results of Analysis

Order	10	20	22	24	26	28	30	32	34	36	38
Number of snarks	1	1	2	0	8	1	11	13	1503	484	≥ 39

Table 4.1: Numbers of irreducible cyclically 5-connected snarks of specific order.

Isaacs Flower Snarks

We describe a non-standard construction of Isaacs flower snarks in the lights of our color-expansion from the Petersen graph. The Flower snark J_3 arose from the Petersen graph by a color-expansion of a vertex to a triangle. Recall that J_3 is a trivial snark. For each $k \geq 2$, the Flower snark J_{2k+1} can be constructed from the snark J_{2k-1} by a color-expansion of Y_2 to Y_4.

Order up to 30

The smallest irreducible cyclically 5-connected snark is the Petersen graph of order 10. It is followed by the flower snark J_5 on 20 vertices. The only irreducible cyclically 5-connected snarks of order 22 are the two Loupekine snarks (see Figure 3.3). Both of them contain the 5-pole P_{NN}, so they can be constructed from the Petersen graph by a color-expansion of P_2 to P_{NN}.

Among the snarks of order 26, there are 8 irreducible cyclically 5-connected snarks. All of them contain the 5-pole P_{NT} and also P_{TT} and hence they are color-reducible, in this case to the Petersen graph. All of them contain the uncolorable 7-pole from Figure 3.5 consisting of one dyad connected to two triads.

There is only one irreducible cyclically 5-connected snark of order 28, it is the Flower snark J_7. Actually, its cyclical connectivity is 6 making it the smallest cyclically 6-connected snark.

Among irreducible cyclically 5-connected snarks, one of them has girth 6, it is the double-star snark described by Isaacs [8]. All other snark of order 30 arose from the Blanuša snark by a color-expansion of P_2 to P_{NN} (6 snarks from type 1 and 4 snark from type 2 of Blanuša snark). Note that this operation increased the cyclically connectivity of the smaller snark.

There are no irreducible snarks of order 12, 14, 16, 24. All irreducible snarks of order 18 are cyclically 4-connected, namely, they are the two Blanuša's snarks.

Order 32

From 13 studied snarks of order 32, 11 snarks contain the 5-pole P_{NN}. All of them can be constructed by a color-expansion from the Flower snark J_5.

Class 32-1

The remaining two snarks consist of three Petersen negators $N_i(I_i, O_i; r_i)$ for $i \in \{1, 2, 3\}$ and one 7-pole M_{11} which are connected as shown in Figure 4.3 while we performed junctions of connector O_2 and I_1, O_1 and I_3. In general, the negators N_1, N_2, N_3 can be taken from arbitrary snark. We explain that the graph G shown in Figure 4.3 is a snark.

Suppose that φ is a coloring of G. Let $a = \varphi(r_1)$. One of the connectors of N_1 has to have zero flow, without loss on generality, let it be the connector I_1. Then $\varphi_*(O_1) = a = \varphi_*(I_3)$. From the negator N_3, we get that $\varphi_*(O_3) = 0$ and $\varphi(r_3) = a$. Now, take a look on the 7-pole M_{11}. The semiedges e_1, e_2 connected to the connector O_3 have the same color and so have the semiedges e_3 and e_4 connected to r_3 and r_1, respectively. From the Parity Lemma, the sum of the flows through the remaining

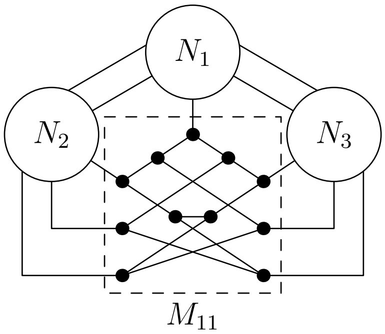

Figure 4.3: The structure of class 32-1 snarks

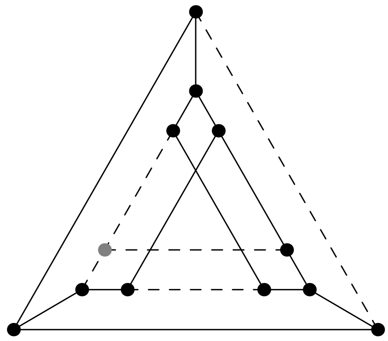

Figure 4.4: The 7-pole M_{11} constructed from J_3

three semiedges e_5, e_6, e_7 is zero. Therefore, we can perform junctions of e_1 and e_2, e_3 and e_4 and add one new vertex incident with e_5, e_6, e_7 giving rise to a graph H which can be colored by a extension of φ. However, the graph H is isomorphic to the flower snark J_3, so we have a contradiction.

Actually, the 7-pole M_{11} contained in the snark G is the flower snark J_3 with one removed vertex and two subdivided links (see Figure 4.4). Moreover, the construction of the snark G uses the symmetry of the 7-pole M_{11} as this 7-pole can be completed to J_3 in two symmetric ways. Thus the 7-pole M_{11} can not be replaced with a 7-pole constructed from arbitrary snark by subdividing two edges and removing one vertex.

Order 34

Among the studied snarks of order 34, 26 of them contain the 5-pole P_{NN} and they can be constructed from the Loupekine snark (both types) by a color-expansion of P_2. The 5-pole P_{NT} is contained in 1084 snarks which arose from the Blanuša snark (both types) by a color-expansion of one vertex and one edge. 84 snarks contain the 5-pole P_{TT} and can be constructed from the Blanuša snark (both types) using the color-expansion of P_2. 72 snarks arose from the Petersen graph by employing superposition described in the section 3.7.

Analyzing the structure of remaining snarks, we observed 6 classes of snarks covering them. Snarks of order 34 contained mostly triads and dyads. It is convenient, as we can identify them with no obstacles. However, the fact that those 5-poles are taken from the Petersen graph is not necessary considering their color properties, therefore we describe discovered six classes using multipoles created from arbitrary snarks yielding into infinite families of snarks.

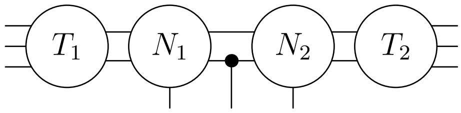

Figure 4.5: A uncolorable 9-pole M_1 contained in class 34-1 snarks.

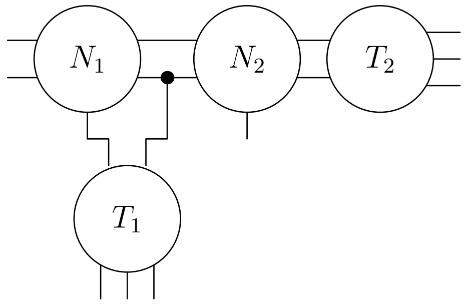

Figure 4.6: A uncolorable 9-pole M_1 contained in class 34-2 snarks.

Class 34-1

We take two negators N_1, N_2 and two proper $(2,3)$-poles T_1, T_2 and construct from them a 9-pole M_1 as shown in Figure 4.5. We claim that M_1 is an uncolorable multipole.

Let φ be a coloring of M_1. Denote the dangling edge which is incident with none of negators and proper $(2,3)$-poles as e. One connector of N_1 is connected to a proper connector of T_1, so the flow through it is non-zero. That means that the other connector O_1 has zero flow. Similarly, the connector I_2 of N_2 not connected to the proper $(2,3)$-pole T_2 has zero flow. If we use the Parity Lemma on the 5-pole between the negators N_1 and N_2, we get $\varphi(e) = \varphi_*(O_1) + \varphi_*(I_2) = 0$ which is a contradiction.

Among the remaining snarks of order 34, there are 21 snarks which contain the described multipole M_1. In all of them, M_1 consists of dyads and triads and has 33 vertices.

Class 34-2

Again, we take two negators N_1, N_2 and two proper $(2,3)$-poles T_1, T_2 and connect them in a 9-pole M_2 as in Figure 4.6. We show that M_2 is a color-closed multipole.

Consider a coloring φ of M_2. Let $O_1 = (e_1, e_2)$ and $I_2 = (e_1, e_3)$ be the connectors of N_1 and N_3 respectively which has a common edge e_1. Denote v the vertex incident with both the edges e_2 and e_3 and f the semiedge incident with v. As one connector of N_2 is connected to a proper $(2,3)$-pole, for the other connector I_2, we have $\varphi_*(I_2) = 0$. For the connector O_1, we have $\varphi_*(O_1) = \varphi(e_1) + \varphi(e_2) = \varphi_*(I_2) + \varphi(f) = \varphi(f)$ and thus the residual edge of N_1 has the color $\varphi(f)$. However for the connector D_1 of size two of T_1, we have $\varphi_*(D_1) = \varphi(f) + \varphi(f) = 0$ contradicting that the $(2,3)$-pole T_1 is proper.

Multipole M_2 contained in studied snarks of order 34 consist of dyads and triads and has 33 vertices. We have identified 90 snarks containing this 9-pole M_2. Note that

54 of snarks of this class contain also the 5-pole P_{NT} and 18 snarks contain the 5-pole P_{TT}.

Class 34-3

Let S be a snark. We remove two adjacent vertices u, v from S resulting in a 4-pole S' with dangling edges f_1, f_2 formerly incident with u and f_3, f_4 formerly incident with v. Furthermore, we subdivide a link e in S' into two dangling edges e_1 and e_2. Denote the constructed $(2,2,2)$-pole as $R(A,B,C)$ with $A = (e_1,e_2)$, $B = (f_1,f_2)$ and $C = (f_3,f_4)$. Take a negator $N(I,O)$ and two proper $(2,3)$-poles $T_1(D_1,E_1)$, $T_2(D_2,E_2)$ Perform the junctions of the connectors D_1 and B, A and I, O and E_2 (see Figure 4.7) resulting in a 9-pole M_3. We claim that M_3 is uncolorable.

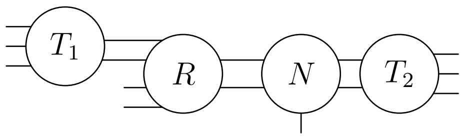

Figure 4.7: A uncolorable 9-pole M_3 contained in class 34-3 snarks.

Suppose that φ is a coloring of the 9-pole M_3. The negator N is connected to the proper $(2,3)$-pole T_2, so $\varphi_*(O) \neq 0$ and $\varphi_*(I) = \varphi_*(A) = 0$. From the proper $(2,3)$-pole T_1 connected to the 6-pole R, we have $\varphi_*(D_1) = \varphi_*(B) = a \neq 0$. From the Parity Lemma for R, we get $\varphi_*(C) = a$.

Now, we can complete R to a snark S by performing a junction of the semiedges e_1, e_2 and adding the vertices u, v. As $\varphi_*(A) = 0$ and $\varphi_*(B) = \varphi_*(C) = a \neq 0$, the coloring φ can be extended to a coloring of the snark S — a contradiction.

There are 72 irreducible 5-connected snarks of order 34 belonging to this class while 18 of these snarks contains also the 5-pole P_{NT}. In all of them, the negator and proper $(2,3)$-poles are taken from the Petersen graph and also the 6-pole R is constructed from the Petersen graph, precisely, it is the double pentagon.

Class 34-4

Have four negators $N_i(I_i,O_i,r_i)$ for $i \in \{1,2,3,4\}$, connect them as shown in Figure 4.8 and denote the resulting graph by G_4. We claim that G_4 is a snark.

Suppose that φ is a coloring of G_4. Without loss on generality, $\varphi_*(O_1) = 0$ and let $\varphi(uw_1) = a \in K$. Then the flow through the connector I_2 is $\varphi_*(I_2) = \varphi_*(O_1) + \varphi(uw_1) = a \neq 0$, so the other connector O_2 has zero flow. In a similar way, we get that $\varphi_*(O_1) = \varphi_*(O_2) = \varphi_*(O_3) = \varphi_*(O_4) = 0$. From the negator N_2, we get that

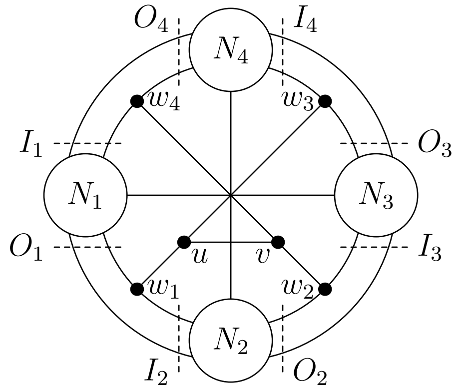

Figure 4.8: Structure of class 36-4 snarks

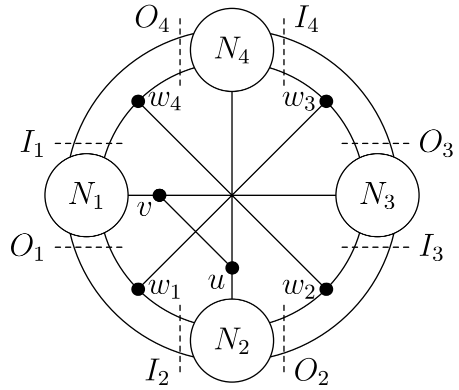

Figure 4.9: Structure of class 36-5 snarks

$\varphi(r_2) = a = \varphi(r_4)$, so also $\varphi_*(I_4) = a$. As $\varphi_*(O_3) = 0$, we get $\varphi(w_3 u) = a$. But the edges $w_3 u$ and $u w_1$ have a common vertex and same color — a contradiction.

We identified 5 snarks having this structure with all four negator taken from the Petersen graph.

Class 34-5

Four negators N_1, N_2, N_3 and N_4 can be arranged in a different way, as shown in Figure 4.9, creating a snark. The proof of its uncolorability is very similar as in the previous class. There are 7 snarks of order 34 having this structure.

Class 34-6

We take three negators N_1, N_2, N_3 and two odd $(2,2,2)$-poles H_1, H_2 and connect them as shown in Figure 4.10 resulting in a graph G_6. We show that G_6 is a snark.

Let φ be a coloring of G_6 and let a be the color of the residual edge of N_1. One of the connectors of N_1 has non-zero flow, without loss on generality, let it be the connector

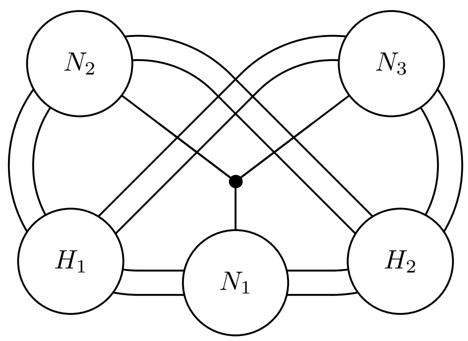

Figure 4.10: Structure of class 36-6 snarks

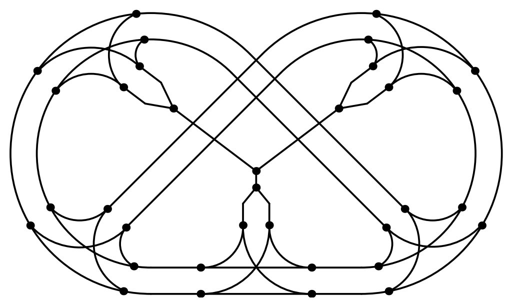

Figure 4.11: Hexagons in a class 34-6 snark

connected to H_1. If we denote the connector of H_1 which is connected to N_i as S_i for $i \in \{1,2,3\}$, then we have $\varphi_*(S_1) = a \neq 0$. The flow through one of the connectors S_2 and S_3 has to be zero. If $\varphi_*(S_2) = 0$, then $\varphi_*(S_3) = a$ from the Parity Lemma. The edges from the S_3 are connected to the connector of N_3 and as their flow is $a \neq 0$, the residual edge of N_3 has the color a, which is a contradiction. If $\varphi_*(S_3) = 0$, we would get a contradiction from the other negator N_2.

Among the studied snarks of order 34, there are two snarks of this type. In both of them, the negators and odd $(2,2,2)$-poles are taken from the Petersen graph, i. e. the odd $(2,2,2)$-poles are hexagons. As the Petersen negator consists of a hexagon with one additional vertex, these two graphs can be redrawn using hexagons instead of negators. One of them is shown in Figure 4.11. Actually, every negator consists of an odd $(2,2,2)$-pole with one additional vertex attached to two semiedges from the same connector, so odd $(2,2,2)$-poles can be found in many other snarks.

To sum up our result, the numbers of studied snarks of order 34 is shown in Table 4.2.

Type of a snark	Number of snarks
Containing P_{NN}	26
Containing P_{NT}	1084
Containing P_{TT}	84
Containing P_{TTT}	22
Superposition	72
Class 34-1	21
Class 34-2	$18 + 18\ (P_{TT}) + 54\ (P_{NT}) = 90$
Class 34-3	$162 + 18\ (P_{NT}) = 180$
Class 34-4	5
Class 34-5	7
Class 34-6	2
TOTAL	1503

Table 4.2: Structure of the irreducible cyclically 5-connected snarks of order 34.

Order 36

From the 484 studied snarks of order 36, 396 snarks contain the 5-pole P_{NT} and all of them arose from the flower snark J_5 by a color-expansion. In 69 snarks, we identified the 5-pole P_{TT} and all of them arose from the J_5 by a color-expansion.

The 5-pole P_{NN} is contained in 10 snarks. They are constructed by color-expansions from two smaller snarks of a order 24 whose structure can be seen in Figure 4.12. Their

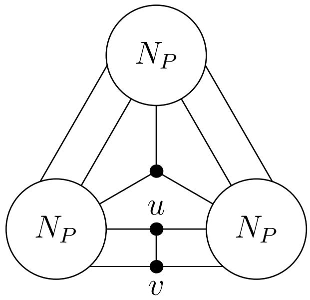

Figure 4.12: A scheme of two cyclically 5-connected reducible snarks of order 24.

structure is similar to Luopekine snarks, they only contain two additional vertices u, v. These two vertices are removable, so these snarks are reducible, although after a color-expansion of P_2 to P_{NN} they become irreducible. In all 10 cases, the color-expanded P_2 contained one of the vertices u, v.

One of the remaining snarks is the flower snark J_9. Now, we take a look on remaining 8 snarks. Among them, we identified two classes.

Class 36-1

Have three negators $N_i(I_i, O_i, r_i)$ for $i \in \{1, 2, 3\}$ and three vertices v_1, v_2 and v_3. Connect them as shown in Figure 4.13. Let e_1, e_2 and e_3 be dangling edges incident with v_1, v_2 and v_3, respectively. Set $I = (r_1, r_2, r_3)$ and $O = (e_2, e_3, e_1)$ and denote the constructed $(3, 3)$-pole as $M_{24}(I, O)$. Take the $(3, 3)$-pole $Y_3(I_Y, O_Y)$ consisting of three copies of the Isaacs $(3, 3)$-pole Y (see Section 3.1). We claim that the $(3, 3)$-poles M_{24} and Y_3 are color-disjoint, in other words that $M_{24} * Y_3$ is a snark.

Suppose the contrary and let φ be a coloring of $M_{24} * Y_3$. Let $\varphi_*(I) = (a, b, c) \in K^3$. For the connectors of the negators, either $\varphi_*(I_1) = \varphi(I_2) = \varphi_*(I_3) = 0$ or $\varphi_*(O_1) = \varphi_*(O_2) = \varphi_*(O_3) = 0$ is held (the proof is similar to the proof in the class 34-4). Consider the first case. Then $\varphi_*(O_1) = \varphi(r_1) = a$ and from the Parity Lemma $\varphi(e_3) = \varphi_*(O_1) + \varphi_*(I_2) = a$. In the same way, we get that $\varphi(O) = (b, c, a)$. In the second case, we get in a very similar way that $\varphi(O) = (c, a, b)$.

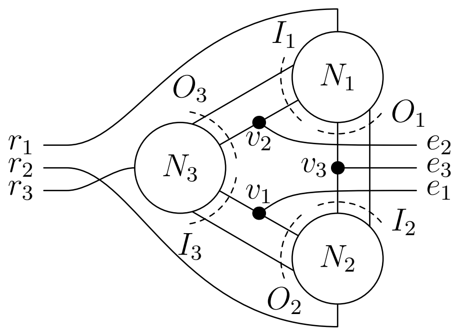

Figure 4.13: A $(3, 3)$-pole M_{24} contained in class 36-1 snarks

As φ is a coloring of Y_3 such that $\varphi(I_Y) = (a, b, c)$, among the colors a, b, c at least two are different. Suppose that two of the colors a, b, c are the same. If two semiedges of the connector I' of the Isaacs $(3, 3)$-pole $Y(I', O')$ have the same color, then the sets of colors used on dangling edges of I' and O' are different. As Y_3 consists of a odd number of the $(3, 3)$-poles Y, the dangling edges from I_Y and O_Y have different sets of colors. However, on the dangling edges from I and O, there is the same set of colors used. This gives us a contradiction.

Suppose that the colors a, b, c are pairwise distinct. In this case, as Y_3 consist of a odd number of $(3, 3)$-poles Y, the triple $\varphi(O_Y)$ is an odd permutation of $\varphi(I_Y)$. However, looking on the connectors of M_{24} we see that $\varphi(O)$ is always an even permutation of $\varphi(I)$ which leads us again to a contradiction.

We identified 6 snarks of order 36 belonging to the class 36-1.

Class 36-2

Let $N_i(I_i, O_i, r_i)$ for $i \in \{1, 2, 3, 4, 5\}$ be five negators. Take a vertex v and connect those negator as shown in Figure 4.14 while we performed the junctions of connectors O_i and I_{i+1} for $i \in \{1, 2, 3, 4\}$. Denote the arisen graph G. We show that G is a snark.

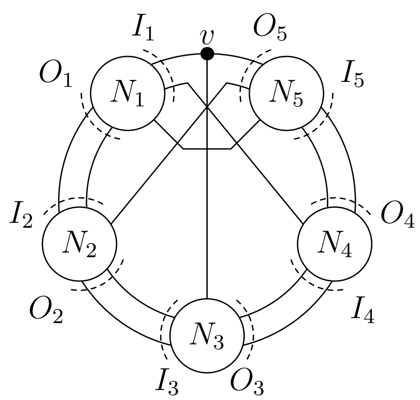

Figure 4.14: Structure of class 36-2 snarks

The graph G is symmetrical, so without loss on generality, say that $\varphi_*(I_3) = 0$. Then $\varphi(r_3) = \varphi_*(O_3) = a$ for some nonzero a. From the negator N_4, we get $\varphi_*(O_4) = 0$, $\varphi(r_4) = a$ and from the negator N_2 we get $\varphi_*(I_2) \neq 0$. That means that in the negator N_1, the connector I_1 has zero flow, so if we denote e the edge from I_1 incident with v, then $0 = \varphi_*(I_1) = \varphi(e) + \varphi(r_4)$ which yields $\varphi(e) = a$. As the edges e and r_3 are adjacent and have a same color, we got a contradiction.

Among the snarks of order 36, we identified 2 snarks of this type.

Order 38

At present, there are 56 known irreducible cyclically 5-connected snarks of order 38. Although, it is very likely that there are more irreducible cyclically 5-connected snarks of order 38, we took a look on those discovered. Among them, we identified the 5-pole P_{NN} in 39 snarks, the 5-pole P_{NT} in 22 snarks and the 5-pole P_{TT} in 7 snarks while there are ten snarks containing both P_{NN} and P_{NT} and six snarks containing both P_{NN} and P_{TT}.

In three snarks, we identified the $(2,2,2;1)$-pole $M_{32} = 3\mathrm{NT}(N_P, N_P, N_P, T_P)$. All three snarks arose from the Petersen graph by a color-expansion of a path of length two and one edge to M_{32}.

Class 38-1

We also introduce one infinite class containing the one remaining snark from the known 56 snarks of order 38. Have four negators $N_i(I_i, O_i; r_i)$ for $i \in \{1,2,3,4\}$ and one proper $(2,3)$-pole $P(D,T)$, connect them as shown in Figure 4.15 and denote the constructed graph as G. We show that G is a snark.

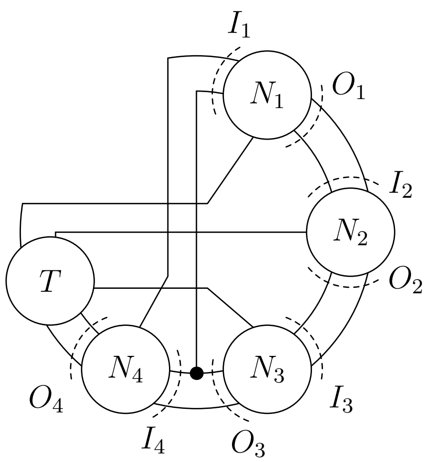

Figure 4.15: Structure of class 38-1 snarks

The connector T is proper, so the edges r_1 and r_2 contained in T have different colors, let us say $\varphi(r_1) = a$ and $\varphi(r_2) = b$ for $a \neq b$. This means that $\varphi_*(O_1) = \varphi_*(I_2) = 0$. Therefore $\varphi_*(I_1) = a$ and $\varphi_*(O_2) = \varphi_*(I_3) = b$. From the properties of the negator N_3, we get that $\varphi(r_3) = b$ and $\varphi_*(O_3) = 0$. Knowing the color of three semiedges of the proper $(2,3)$-pole T, we can from the Parity Lemma determine the flow through the remaining two semiedges, identical to the flow through the connector O_4, as $\varphi_*(O_4) = a+b+b = a$. Then from the negator N_4, we have $\varphi_*(I_4) = 0$. However, from the Parity Lemma, we get a zero flow through the edge $e \in I_1$ different from r_4 which is a contradiction.

In the end, we summarize our results in Table 4.3.

Order	10	20	22	24	26	28	30	32	34	36	38
NN expansion	0	0	2	0	0	0	10	11	26	10	≥ 39
TT expansion	0	0	0	0	8	0	0	0	84	69	≥ 3
NT expansion	0	0	0	0	8	0	0	0	1084	396	≥ 17
TTT expansion	0	0	0	0	0	0	0	0	22	0	≥ 0
Superposition	0	0	0	0	0	0	0	0	72	0	≥ 0
Other	1	1	0	0	0	1	1	2	215	9	≥ 4
TOTAL	1	1	2	0	8	1	11	13	1503	484	≥ 56

Table 4.3

4.3 Further Attempts

In our analysis, we discovered several multipoles, which can be used in a construction of cyclically 6-edge connected snarks.

Take a look on an odd $(2,2,2)$-pole. The $(2,2,2)$-pole V consisting of three vertices with one common neighbor (see the section 2.3) is color-disjoint to each odd $(2,2,2)$-pole. However, as the semiedges in each connector of V are incident with the same vertex, this often leads to cyclical connectivity only 5. This can be seen e. g. in the Petersen graph, where is V connected to a hexagon, in the snarks of class 34-6 or Loupekine snarks, where the Petersen negator can be imagined as hexagon. The smallest known cyclically 6-connected snark where is the 6-pole V connected to an odd $(2,2,2)$-pole is of order 118 and was described by Kochol [9]. There is a question if there exist an odd $(2,2,2)$-pole h of order smaller than 114 and color-disjoint with the $(2,2,2)$-pole V.

Attempts in constructing a cyclically 6-connected snarks can be made from the other point of view. We can find a $(2,2,2)$-pole M color-disjoint to the hexagon, the smallest odd $(2,2,2)$-pole. For the sought $(2,2,2)$-pole M, it is sufficient to be proper as the flow through at least one of the connectors of every odd $(2,2,2)$-pole is zero. However, the 6-pole M has to ensure cyclically 6-connectivity when connected to the hexagon. The $(2,2,2)$-pole V itself has desired coloring properties but produces a 5-edge cut as the dangling edges from each connector are adjacent. Other 6-pole with the desired coloring properties is the $(2,2,2)$-pole P_{TTT} of order 28, which is, unfortunately, only cyclically 5-edge connected.

We employ the Kochol's superposition [10] on the cycle of length 12, marked with bold line in Figure 4.16, where we replace each vertex with the supervertex B and each edge by a proper superedge constructed from J_5 (see Section 3.7). Doing so, we

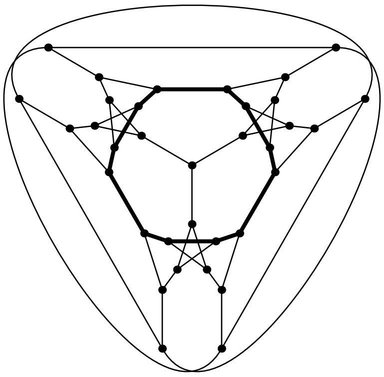

Figure 4.16: A sanrk of order 34 used in the construction of cyclically 6-connected snark of order 250 containig a proper $(2, 2, 2)$-pole

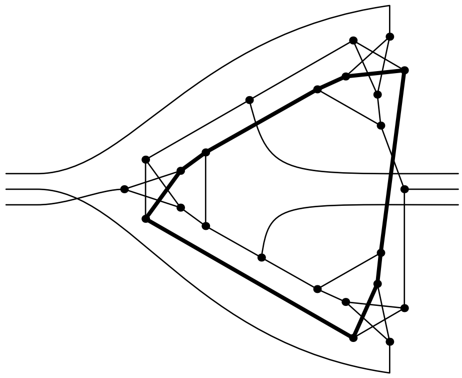

Figure 4.17: A $(3, 3)$-pole M_{24} used in the construction of the $(3, 3)$-pole M_{186} without five-cycles colordisjoint to Y_3

remove from P_{TTT} all 5-cycles resulting into a proper $(2, 2, 2)$-pole V_{244} of order 244 which is color-contained in P_{TTT}. Performing a junction of V_{244} and the hexagon, we get a snark of order 250 (see Figure 4.16, the replaced cycle is marked with a bold line) with cyclical connectivity 6 which was determined using a computer program. Again, there arises a request to find a smaller proper $(2, 2, 2)$-pole whose dangling edges from each connector are not incident with a common vertex.

In the class 36-2 we observed a $(3, 3)$-pole M_{24} color-disjoint with the $(3, 3)$-pole Y_3. Again, the 6-pole M_{24} contains five-cycles and it is only cyclically 5-connected. Employing the Kochol's superposition on the cycle of length 9 we can remove all five-cycles from M_{24} and get a $(2, 2, 2)$-pole M_{186} (see Figure 4.17, the replaced 9-cycle is marked with a bold line) color-contained in M_{24} and therefore color-disjoint to Y_3. The snark $M_{186} * Y_3$ of order 198 is cyclically 6-connected which we determined using a computer.

Conclusion

In our work, we described several ways of constructing snarks. We introduced pairs of multipoles which can be replaced in snarks. It allow us to construct new snarks from a given one or reduce a given snark to a smaller one. Also, we constructed several infinite classes of snarks.

Using discovered operations and classes, we analyzed all 2024 irreducible cyclically 5-connected snarks up to order 36 and some of the order 38, described their structure and explained why they are uncolorable.

We have seen that many of 5-connected snarks are constructed mostly from negators and proper $(2, 3)$-poles. In several snarks, we have found also the multipoles with more than five dangling edges (e. g. class 32-1, 34-3), however in these snarks, the negators and proper $(2, 3)$-poles were the essence of uncolorability, as they forced inadmissible coloring of the dangling edges of the mentioned multipole.

In some classes, we used methods different from flows through connectors in proving uncolorability. In the class 32-1, we used the symmetry of the 7-pole M_{11} and in the class 36-1, we looked at parity of permutations of the colors in the connectors of the $(3, 3)$-pole M_{24}.

Also, there appeared problems for further research. Although, we described some infinite families of snarks, we do not know much about their properties. As we discovered them in the study of irreducible snark, there arises a natural need to find some sufficient condition which would ensure the irreducibility of described classes. It might be sufficient that all negators and proper $(2, 3)$-poles are taken from irreducible snarks. As we have seen among snarks of order 36, this condition is not necessary. Its sufficiency remains open for further research.

Another way of research is to take found multipoles with 6 and more dangling edges and attempt to construct some cyclically 6-connencted snarks. At present, the known smallest cyclically 6-connected snark different from the flower snarks has 118 vertices. However, studying flows through connectors of multipoles will likely be not sufficient as we could observed in several classes of snarks.

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Appendix

We attach a CD with detailed results of our analysis and used computer program.