Digital signature schemes

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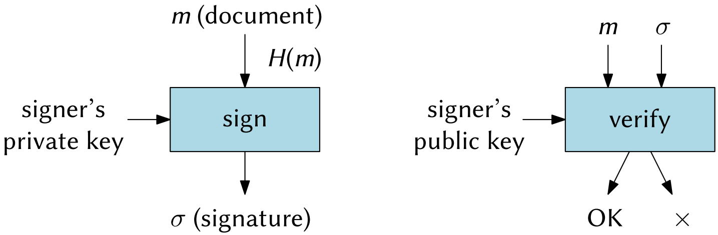
Cryptology 1 (2023/24)

Content

Introduction

- “electronic signatures” in legislation
- “digital signatures” in cryptology
- objectives:
  - authenticity and integrity of signed data
  - non-repudiation of origin
  - (usually) universal verifiability, i.e. anyone can verify the signature
  - unforgeability, efficiency etc.
- objectives impossible to satisfy by a digital signature scheme alone
  - PKI, laws etc. (out of the scope of this lecture)

Digital signature scheme (1)



- asymmetric construction
  - private key – signing
  - public key – verification

Digital signature scheme (2)

- digital signature scheme: $(\mathrm{Gen}, \mathrm{Sig}, \mathrm{Vrf})$
- Gen – PPT algorithm, produces public and private key pair $(\mathrm{pk}, \mathrm{sk})$
- Sig – PPT algorithm, produces a signature from a message and signer's private key: $\sigma = \mathrm{Sig}_{\mathrm{sk}}(m)$
- Vrf – usually deterministic PT algorithm; input: message, signature and signer's public key; $\mathrm{Vrf}_{\mathrm{pk}}(m, \sigma) \in \{\mathrm{true/OK}, \mathrm{false}/\times\}$
- correctness of the scheme:

$$\forall(\mathrm{pk}, \mathrm{sk}) \leftarrow \mathrm{Gen}(1^k)\ \forall m : \mathrm{Vrf}_{\mathrm{pk}}(m, \mathrm{Sig}_{\mathrm{sk}}(m)) = \mathrm{true}$$

Digital signature schemes – remarks

- schemes with appendix
  - the most common type
  - original document needed for the signature verification
- schemes with message recovery
  - verification produces from a signature the original message and some additional data to verify its correctness
  - rarely used
- this lecture – schemes with appendix
- reasons for using hash function in digital signature schemes
  - shorter, fixed-length data for signing
  - preventing certain attacks, e.g. random message forgery (see later)
- using h.f. $\Rightarrow$ the security depends on h.f. properties, e.g. collision resistance

Security of digital signatures

- various possibilities; as usual: use the strongest definition
- idea similar to MAC security
- attacker has access to a public key
- EUF-CMA
  - CMA (chosen message attack) – the attacker has access to $\mathrm{Sig}_{\mathrm{sk}}(\cdot)$ oracle
  - EUF (existential unforgeability) – the attacker tries to create a message m (not previously queried) and a valid signature σ, i.e. $\mathrm{Vrf}_{\mathrm{pk}}(m, \sigma) = \mathrm{true}$
- scheme is EUF-CMA secure if the success probability of any PPT attacker is negligible

RSA signature scheme

- RSA instance/parameters as before:
  - public key: (e, n)
  - private key: d
  - all optimizations can be applied
- 1st attempt (without hashing):
  - Sig: $\sigma = m^d \bmod n$
  - Vrf: $\sigma^e \bmod n = m\,?$
  - correctness follows from the properties of RSA
- problems:
  - only for short messages
  - random message forgery: $(\underbrace{\sigma^e \bmod n}_{m}, \sigma)$ for $\sigma \in \mathbb{Z}_n$

    the attacker has no control over the message value
  - another forgery (using homomorphic property of RSA): take two valid pairs (m_1, σ_1), (m_2, σ_2), and produce $(m_1 m_2 \bmod n, \sigma_1 \sigma_2 \bmod n)$

RSA signature scheme – standard “textbook” version

- 2nd attempt (with hashing):
  - Sig: $\sigma = H(m)^d \bmod n$
  - Vrf: $\sigma^e \bmod n = H(m)$?
- properties:
  - messages of arbitrary length
  - H is preimage resistant (infeasible to invert) $\Rightarrow$ prevents random message forgery
- H should be collision resistant
- FDH (Full Domain Hash) signature scheme using H with image $\mathbb{Z}_n$
  - EUF-CMA secure in random oracle model (for H), assuming the hardness of the RSA problem
- $H(m)$ usually shorter than $n \Rightarrow$ padding
  for randomization and (sometimes) provable security

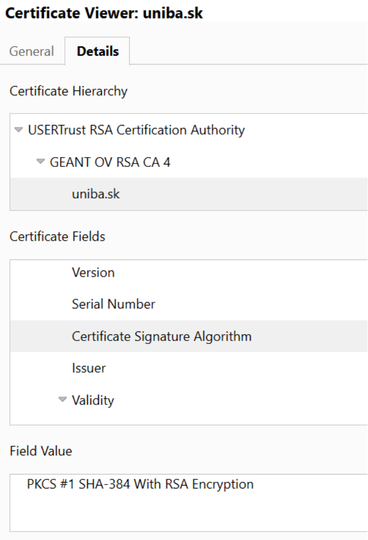
PKCS #1 v1.5

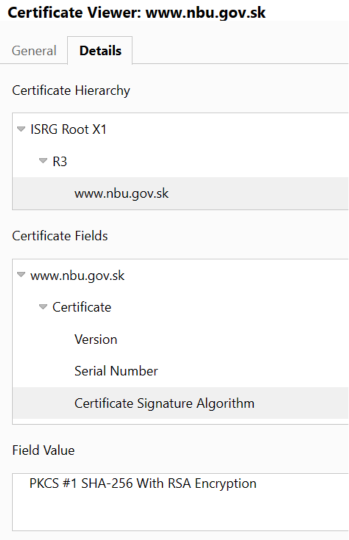
- construction standardized in 1998
- padded digest $H(m)$:

$$0x00 \,\|\, 0x01 \,\|\, 0xff \,\|\, \ldots \|\, 0xff \,\|\, 0x00 \,\|\, H(m)$$

  "Moreover, while no attack is known against the EMSA-PKCS-v1_5 encoding method, a gradual transition to EMSA-PSS is recommended as a precaution against future developments." (RFC 8017, 2016)
- frequently used in practice, e.g. X.509 certificates:
  - "sha256RSA" or "PKCS #1 SHA-256 With RSA Encryption" signature algorithm
- proof of PKCS #1 security (2018), EUF-CMA in RO model under the standard RSA assumption

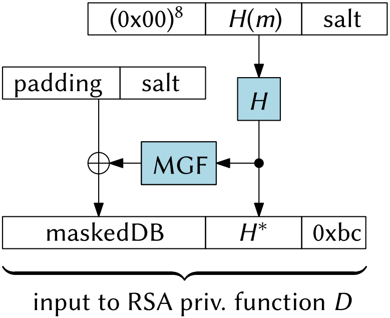
PKCS #1 v1.5 examples





RSA-PSS

- Probabilistic Signature Scheme, PKCS #1 v2.2 (RFC 8017)
- provable security in random oracle model



- salt – sequence of random bytes
- padding = 0x00 || … || 0x00 || 0x01
- MGF – mask generation function (used in OAEP as well)

RSA-PSS – verify

$\mathrm{Vrf}_{\mathrm{pk}}(m, \sigma)$:

1. parse and verify: $\sigma^e \bmod n \mapsto \mathrm{maskedDB} \parallel H^* \parallel \mathrm{0xbc}$
2. $\mathrm{DB} = \mathrm{maskedDB} \oplus \mathrm{MGF}(H^*)$
3. parse and verify: $\mathrm{DB} \mapsto \mathrm{padding} \parallel \mathrm{salt}$
4. verify that $H^* = H((\mathrm{0x00})^8 \parallel H(m) \parallel \mathrm{salt})$

the signature is correct if all verifications succeed

ElGamal signature scheme

- T. ElGamal (1984)
- it is impossible to use ElGamal encryption scheme's algorithms for digital signatures
  - encryption is not a function (randomized …a good thing)
  - very few schemes offer bijections like RSA
  - specific signature scheme must be designed
- initialization identical to encryption scheme: $\mathrm{pk} = (p, g, y)$, $\mathrm{sk} = x$
  - $y = g^x \bmod p$
  - let g be a generator of $(\mathbb{Z}_p^*, \cdot)$
  - scheme can be "rephrased" in other groups
- $\mathrm{Sig}_{\mathrm{sk}}(m) = (r, s)$:
  1. $k \xleftarrow{\$} \mathbb{Z}_{p-1}$ such that $\gcd(k, p-1) = 1$
  2. $r = g^k \bmod p$
  3. $s = (H(m) - xr) \cdot k^{-1} \bmod (p-1)$

ElGamal signature scheme – verification and correctness

- $\mathrm{Vrf}_{\mathrm{pk}}(m, (r, s))$: correct if

$$1 \leq r < p \qquad \& \qquad y^r \cdot r^s \equiv g^{H(m)} \pmod{p}$$

- correctness:
  - the first part is trivial
  - the second part: $y^r \cdot r^s \equiv g^{xr} \cdot g^{ks} \equiv g^{xr+ks} \equiv g^{H(m)} \pmod{p}$
- efficiency
  - Sig – single modular exponentiation (can be precomputed)
  - Vrf – 3 modular exponentiations
  - signature's length $\sim$ a pair from $\mathbb{Z}_p \times \mathbb{Z}_p$

ElGamal – security (1)

- computing x from y is a discrete logarithm problem
- predictable (leaked) k results in private key compromise:

$$s = (H(m) - xr) \cdot k^{-1} \bmod (p-1) \quad \Rightarrow \quad x = (H(m) - ks)r^{-1} \bmod (p-1)$$

- test $1 \leq r < p$ is necessary; let us assume verification without the test:
  - let (r, s) be a signature for m, i.e. $g^{H(m)} \equiv y^r \cdot r^s \pmod{p}$
  - we compute a signature (r', s') for some $m' \neq m$

1. $u = H(m') \cdot H(m)^{-1} \bmod (p-1)$ (assuming that $H(m)$ is coprime to $p-1$)

$$g^{H(m')} \equiv g^{H(m)u} \equiv y^{ur} \cdot r^{us} \pmod{p}$$

2. we set $s' = us \bmod (p-1)$ and compute r' satisfying

$$r' \equiv ru \pmod{p-1}$$
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  - apply CRT; with overwhelming probability $r' \geq p$, otherwise $u = 1$ and we have a collision in H: $H(m') \equiv H(m) \pmod{p-1}$

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ElGamal – security (2)

- Bleichenbacher's attack (1996)
  - forging signatures if g has only small factors and $g \mid (p-1)$
  - e.g. $g = 2$ is a bad choice
  - Remark: for discrete logarithm problem all generators are equivalent

- reusing k (for two distinct messages):
  - $m_1, (r, s_1) \implies H(m_1) \equiv xr + ks_1 \pmod{p-1}$
    $m_2, (r, s_2) \implies H(m_2) \equiv xr + ks_2 \pmod{p-1}$
  - we have $H(m_1) - H(m_2) \equiv k(s_1 - s_2) \pmod{p-1} \qquad (\star)$
  - let $d = \gcd(s_1 - s_2, p-1)$
  - if $d = 1$ then $k = (H(m_1) - H(m_2))(s_1 - s_2)^{-1} \bmod (p-1)$
  - otherwise we divide the equation $(\star)$ by d, solve it mod $(p-1)/d$, and then test d candidates for k
  - having k we can easily find the private key x

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ElGamal – security (3)

- random message forgery (when H is not used)

1. $i, j \xleftarrow{\$} \mathbb{Z}^*_{p-1}$
2. $r = g^i \cdot y^j \bmod p$
3. $s = -r \cdot j^{-1} \bmod (p-1)$
4. $m = s \cdot i \bmod (p-1)$

correctness:

$$\begin{aligned} y^r \cdot r^s &\equiv y^r \cdot g^{is} \cdot y^{js} \\ &\equiv y^r \cdot g^{is} \cdot y^{-jrj^{-1}} \\ &\equiv g^{is} \equiv g^m \pmod p \end{aligned}$$

Digital Signature Algorithm (DSA)

- removed from FIPS 186-5 (2023) – RSA, ECDSA, EdDSA
  - other alternatives available, e.g. ECDSA is faster, with shorter keys
- initialization:
  1. generate primes p, q (e.g. $|p| = 2048$, $|q| = 256$) such that $q \mid (p-1)$
  2. generate $h \xleftarrow{\$} \mathbb{Z}_{p-1}$ such that $g = h^{(p-1)/q} > 1$
     g is a subgroup generator (order q)
     public domain parameters: (p, q, g)
  3. private key: $x \xleftarrow{\$} \mathbb{Z}_q^*$
  4. public key: $y = g^x \bmod p$

DSA – signing and verification

- $\mathrm{Sig}_{\mathrm{sk}}(m)$:
  1. $r = g^k \bmod p \bmod q$, where $k \xleftarrow{\$} \mathbb{Z}_q^*$
  2. $s = k^{-1}(H(m) + xr) \bmod q$
  3. if $r = 0$ or $s = 0$ start again with step 1 (extremely unlikely)
  4. $\sigma = (r, s)$

- $\mathrm{Vrf}_{\mathrm{pk}}(m, (r, s))$:
  1. verify that $r, s \in \mathbb{Z}_q^*$
  2. $u_1 = H(m) \cdot s^{-1} \bmod q$
     $u_2 = r \cdot s^{-1} \bmod q$
  3. verify that $(g^{u_1} \cdot y^{u_2} \bmod p) \bmod q = r$

- correctness:

$$(g^{u_1} \cdot y^{u_2} \bmod p) \bmod q = g^{H(m)s^{-1} + xrs^{-1}} \bmod p \bmod q$$
$$= g^{s^{-1}(H(m)+xr)} \bmod p \bmod q = g^k \bmod p \bmod q = r$$

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  3. verify that $(g^{u_1} \cdot y^{u_2} \bmod p) \bmod q = r$
- correctness:

$$
\begin{aligned}
(g^{u_1} \cdot y^{u_2} \bmod p) \bmod q &= g^{H(m)s^{-1} + xrs^{-1}} \bmod p \bmod q \\
&= g^{s^{-1}(H(m)+xr)} \bmod p \bmod q = g^k \bmod p \bmod q = r
\end{aligned}
$$

DSA – remarks

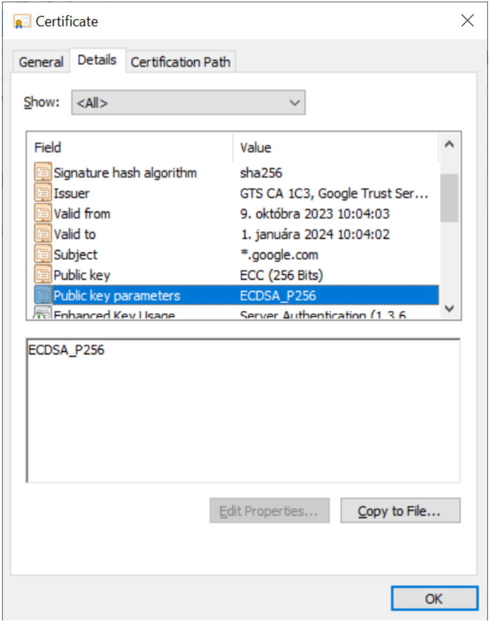
- if $r = 0$ then the signature does not depend on x
- if $s = 0$ then $s^{-1} \bmod q$ does not exist
- efficiency: shorter signatures (comparing to ElGamal's), faster than ElGamal (shorter exponents), r can be precomputed
- H required to prevent random message forgery (try yourself)
- the parameters p, q and g can be shared
  - rare for DSA; ensure that parameters are not maliciously prepared
  - verifiable procedure for generating the parameters (part of the standard)
  - ECDSA: fixed curves and parameters are used (approved)

ECDSA

- point G – generator of subgroup of prime order n
- private key: $d \xleftarrow{\$} \mathbb{Z}_n^*$; public key: $Q = dG$ and domain parameters
- simplifying some details (e.g. conversions bitstring $\leftrightarrow$ integer)
- $\mathrm{Sig}_{\mathrm{sk}}(m)$:
  1. $(x, y) = kG$, where $k \xleftarrow{\$} \mathbb{Z}_n^*$
  2. $r = x \bmod n$
  3. $s = k^{-1}(H(m) + dr) \bmod n$
  4. if $r = 0$ or $s = 0$ start again with step 1 (extremely unlikely)
  5. $\sigma = (r, s)$
- $\mathrm{Vrf}_{\mathrm{pk}}(m, (r, s))$:
  1. verify that $r, s \in \mathbb{Z}_n^*$
  2. $u_1 = H(m) \cdot s^{-1} \bmod n$
     $u_2 = r \cdot s^{-1} \bmod n$
  3. $(x, y) = X = u_1 G + u_2 Q$ (if $X = 0$ reject)
  4. accept iff $x \bmod n = r$
- correctness – obvious

ECDSA – remarks

- shorted keys and faster (comparing to DSA)
- somewhat popular (usually with P-256 curve), 12% according to the CT logs
- used with Bitcoin (curve Secp256k1)



DSA/ECDSA – problems with k

- predictable/repeating k results in private key compromise
  - sometimes it hurts: (2010) Sony PS3 ECDSA with constant k, (2013) Android's Java SecureRandom with low entropy
  - variant with deterministic (EC)DSA proposed in RFC 6979
- various attacks when some additional information about k is known, for example (Faugère et al., 2012):
  - assumption: known messages and signatures, such that k-values share some common bits (the bits themselves are unknown to the attacker)
  - works for both DSA and ECDSA
  - concrete results for 160-bit q:
    100 signed messages with shared 4 LSBs … 100% probability of success
    200 signed messages with shared 3 LSBs … 100% probability of success
    400 signed messages with shared 1 LSBs … 90% probability of success

Schnorr signature scheme

- simple construction, base for other schemes
- let us use the DSA's parameters (any underlying group can be used):
  - public domain parameters: (p, q, g)
  - g is a generator of some subgroup (order q)
  - private key: $x \xleftarrow{\$} \mathbb{Z}_q^*$
  - public key: $y = g^x \bmod p$
- hash function H with image $\mathbb{Z}_q$
- $\mathrm{Sig}_{\mathrm{sk}}(m)$:
  1. $r = H(m \,||\, g^k \bmod p)$, where $k \xleftarrow{\$} \mathbb{Z}_q^*$
  2. $s = k + xr \bmod q$
  3. $\sigma = (r, s)$
- $\mathrm{Vrf}_{\mathrm{pk}}(m, (r, s)) = \text{true} \iff H(m \,||\, g^s \cdot y^{-r} \bmod p) = r$
- correctness: $g^s \cdot y^{-r} \equiv g^{k+xr} \cdot g^{-xr} \equiv g^k \pmod{p}$

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Schnorr signature scheme – security

- EUF-CMA in ROM under the discrete logarithm assumption
- again: k must be unpredictable

EdDSA

- EdDSA (Edwards Curve Digital Signature Algorithm, RFC 8032)
  - deterministic variant of Schnorr signature scheme
  - included in the FIPS 186-5 (Ed448 and Ed25519)
- Ed25519 ~ EdDSA with Curve25519 (in a different form) and SHA-512
  - optimized for speed and security
- Simplified EdDSA – parameters:
  - H – hash function with $2b$-bit output
  - B – point on elliptic curve, that generates a subgroup of prime order l
- Keys:
  - b-bit string k
  - compute $H(k) = \mathbf{h} = (h_0, \dots, h_{2b-1})$
  - left half of $\mathbf{h}$ is a scalar $a = \mathbf{h}[0 \dots b-1]$
  - $A = aB$
  - private key: k (sometimes with A) or a with the right half $\mathbf{h}[b \dots 2b-1]$
  - public key: A

EdDSA – signing and verification

- Signing a message m:
  1. $r = H(\mathbf{h}[b \dots 2b-1], m)$
  2. $R = rB$
  3. $s = r + H(R, A, m) \cdot a \bmod l$
  4. signature: (R, s)
- Verification of (R, s) for m:
  - check if $sB = R + H(R, A, m) \cdot A$
  - correctness:

$$sB = (r + H(R, A, m) \cdot a)B = rB + H(R, A, m) \cdot (aB) = R + H(R, A, m) \cdot A$$